STEP Exercises - Integers (Sol'ns) (5 pages; 12/9/18)

(1) Can n^3 equal $n + 12345670$ (where n is a positive integer)?

Solution

Rearrange to $n^3 - n = 12345670$

$n^3 - n = n(n^2 - 1) = n(n-1)(n+1)$, and one of these factors must be a multiple of 3; whereas 12345670 is not a multiple of 3 (since $1 + 2 + 3 + 4 + 5 + 6 + 7 + 0$ isn't a multiple of 3); so answer is No.

(2) Find all positive integer solutions of the equation

$xy - 8x + 6y = 90$

Solution

[Aiming for something of the form $f(x)g(y) = c$, where c is an integer:]

$xy - 8x + 6y = (x+6)(y-8) + 48$,

so that the original equation is equivalent to

$(x+6)(y-8) = 42$

The positive integer solutions are given by:

$x + 6 = 7, y - 8 = 6$

$x + 6 = 14, y - 8 = 3$

$x + 6 = 21\ y - 8 = 2$

$x + 6 = 42, y - 8 = 1$,

so that the solutions are:

$x = 1, y = 14$

$x = 8, y = 11$

$x = 15, y = 10$

$x = 36, y = 9$

(3) Show that $f(h) = h\left(\frac{N}{N+1}\right)^h$ is maximised when $h = N$ (h is an integer).

Solution

Show that $\frac{f(N)}{f(N-1)} > 1$ and $\frac{f(N)}{f(N+1)} > 1$

(4) **D.** The smallest positive integer n such that

$$1 - 2 + 3 - 4 + 5 - 6 + \cdots + (-1)^{n+1}n \geqslant 100,$$

is

(a) 99, (b) 101, (c) 199, (d) 300.

Solution

In order to simplify the last term on the LHS, we could consider separately the cases of n being even and odd.

With even n, the LHS becomes $1 - 2 + 3 - 4 + \cdots - 2m$, writing $n = 2m$.

By grouping the terms as $(1 - 2) + (3 - 4) + \cdots - 2m$, we see that this has a negative value.

So n must be odd, and the LHS becomes

$1 - 2 + 3 - 4 + \cdots + (2m + 1)$, writing $n = 2m + 1$

And the terms can be grouped to give

$(1-2)+(3-4)+\cdots([2m-1]-2m)+(2m+1)$

$= m(-1)+(2m+1) = m+1$

So we want $m+1 \geq 100$, and hence

$n = 2m+1 \geq 2(99)+1 = 199$

Ans: (c)

(5)

J. The number of *pairs* of *positive integers* x, y which solve the equation

$$x^3 + 6x^2y + 12xy^2 + 8y^3 = 2^{30}$$

is

(a) 0, (b) 2^6, (c) $2^9 - 1$, (d) $2^{10} + 2$.

Solution

Rearrange the LHS. The presence of $8y^3$ suggests $(x+2y)^3$.

Having obtained $x + 2y = 2^{10}$, we can simplify matters by writing $x = 2u$ (since x has to be even), to give $u + y = 2^9$.

Then y can take the values $1, 2, \ldots, 2^9 - 1$ (with $x = 2^{10} - 2y$), so that there are $2^9 - 1$ such pairs.

Ans. is (c)

(6) **J.** Let a, b, c be positive numbers. There are *finitely* many *positive whole* numbers x, y which satisfy the inequality

$$a^x > c\,b^y$$

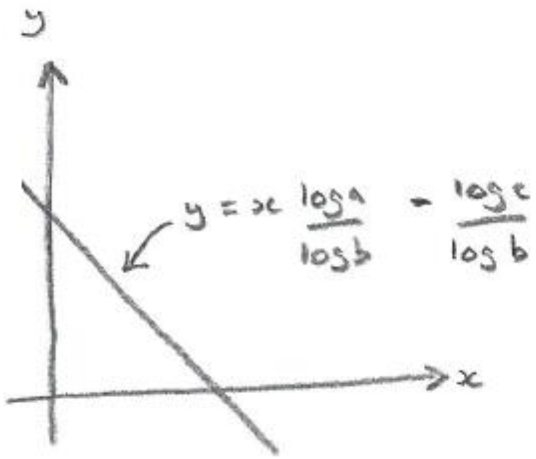
if

(a) $a > 1$ or $b < 1$.
(b) $a < 1$ or $b < 1$.
(c) $a < 1$ and $b < 1$.
(d) $a < 1$ and $b > 1$.

Solution

Take logs of both sides.

Here we require the gradient of the line $y = x\frac{loga}{logb} - \frac{logc}{logb}$ to be negative (to give a finite number of integer pairs (x, y)), and this is only satisfied by (d). (The log can be to any base of course.)



(7) **I.** The function $S(n)$ is defined for positive integers n by

$$S(n) = \text{sum of the digits of } n.$$

For example, $S(723) = 7 + 2 + 3 = 12$. The sum

$$S(1) + S(2) + S(3) + \cdots + S(99)$$

equals

(a) 746, (b) 862, (c) 900, (d) 924.

Solution

The official (MAT) solution is based on spotting the 'lateral thinking' idea that, by symmetry, there must be 20 of each of the 10 digits

0, 1, 2, ..., 9 amongst the numbers 00, 01, ..., 99 (there being 200 digits in total).

Thus the required sum is:

$$20(0 + 1 + \cdots + 9) = 20\left(\frac{1}{2}\right)(9)(10) = 900$$

Alternative method:

$$S(1) + \cdots + S(9) = \sum_{i=1}^{9} i = \frac{1}{2}(9)(10) = 45$$

$$S(10) + \cdots + S(19) = (10 \times 1) + \sum_{i=1}^{9} i = 10 + 45$$

[not writing 55 at this stage, so as not to lose the 45, which is clearly going to be cropping up again]

$$S(20) + \cdots + S(29) = (10 \times 2) + \sum_{i=1}^{9} i = 20 + 45\,,$$

and so on, giving a grand total of

$$(10 + 20 + \cdots + 90) + (10 \times 45) = 10(45) + 450 = 900$$

Ans: (c)