STEP Exercises - Transformations (sol'ns)

(5 pages; 18/9/18)

(1) Suppose that $y = f(x)$ is reflected in the line $x = a$, to give $y = f(u)$. Find u in terms of x.

Solution

(i) A particular point can be reflected in the line $x = a$ by considering a translation of both line and point by an amount a to the left, then performing a reflection in the y-axis and translating everything back, by a to the right.

In mathematical terms, x is first of all replaced by $x + a$; then x is replaced by $-x$, and finally x is replaced by $x - a$.

Thus $f(x) \rightarrow f(x + a) \rightarrow f(-x + a) \rightarrow f(-[x - a] + a) =$
$f(2a - x)$

[As an aid to memory, consider the reflection of $y = sinx$ about $x = \frac{\pi}{2}$, which is $y = \sin(\pi - x)$]

(2) What combination of transformations converts $y = 2^x$ to

$y = 2^{4x-2}$?

Solution

$y = 2^x \rightarrow y = 2^{4x}$ is a stretch of scale factor $\frac{1}{4}$ in the x-direction

Then $y = 2^{4x} \rightarrow y = 2^{4(x-\frac{1}{2})} = 2^{4x-2}$ is a translation of $\begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$

[Alternatively, $y = 2^{4x} \rightarrow y = \left(\frac{1}{4}\right) 2^{4x} = 2^{4x-2}$ is a stretch of scale factor $\frac{1}{4}$ in the y-direction.]

(3) What happens to the graph of $y = f(x)$ when it is transformed to:

(a) $y = f(|x|)$ (b) $|y| = f(x)$

Solution

(a) When $x \geq 0$, $f(|x|) = f(x)$; when $x < 0$, $f(|x|) = f(-x)$; ie that part of $y = f(x)$ to the right of the y-axis is reflected in the y-axis.

So $y = f(|x|)$ is the right half of $y = f(x)$, together with its reflection in the y-axis.

(b) First of all, $|y| = f(x)$ is only defined for x such that $f(x) \geq 0$.

The graph of $|y| = f(x)$ is similar to that of $y^2 = f(x)$, or

$y = \pm\sqrt{f(x)}$, in that it has two branches: $y = f(x)$ and

$y = -f(x)$.

So, provided $f(x) \geq 0$, $|y| = f(x)$ is the same as $y = f(x)$, with the addition of its reflection in the x-axis.

(4) Sketch the following:

(i) $y = ln(1-x)$

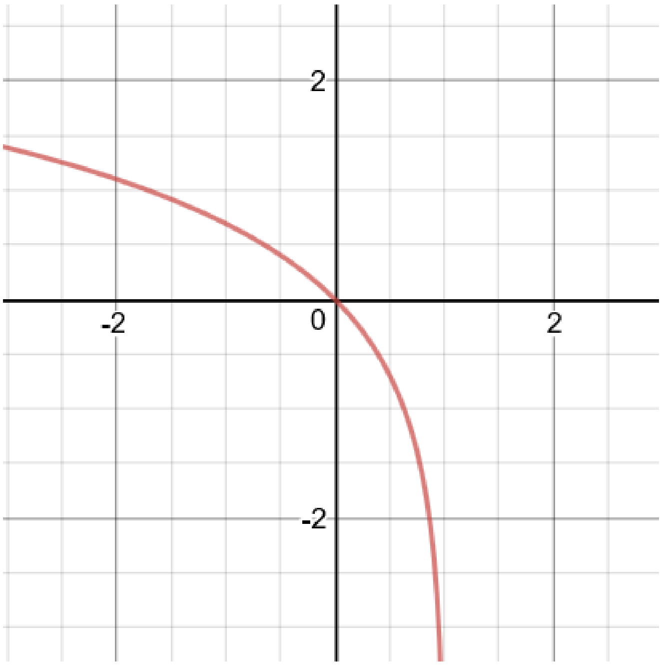
(ii) $y = ln(x^2 - 1)$

(iii) $y = ln|x^2 - 1|$

Solution

(i) $y = \ln(1-x)$ is the reflection in $x = \frac{1}{2}$ of $y = lnx$

[$y = lnx \rightarrow y = \ln(-x)$ is a reflection in the y-axis (note that the domain changes to negative x); then $\ln(-x) \rightarrow \ln(-[x-1]) = \ln(1-x)$ is a translation of 1 to the right, which can be seen to be a reflection in $x = \frac{1}{2}$; also, compare with $y = \sin(\pi - x)$, which is the reflection in $x = \frac{\pi}{2}$ of $y = sinx$]



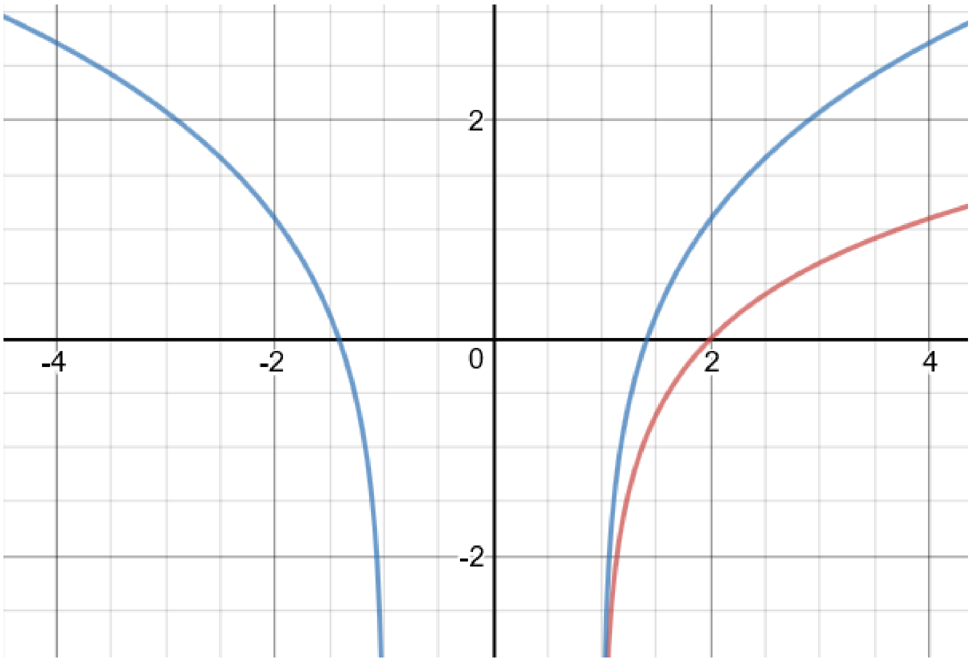
$y = \ln(1-x)$

(ii) $y = \ln(x^2 - 1)$ is an even function; ie it's symmetric about the y-axis. It is undefined for $-1 \leq x \leq 1$

The right-hand branch can be obtained from $y = \ln(x-1)$: For $x > 1$, $y = f(x^2)$ will be a compressed version of $y = f(x)$, with equality as $x \rightarrow 1$ [eg to obtain the point $(2, f(2^2))$, we start at

$(2,0)$ on the x-axis, then look to the right to obtain $(2^2, 0)$, then up to the curve $y = f(x)$, to find the point $(2^2, f(2^2))$, which we drag to the left, to give $(2, f(2^2))$; thus the process is similar to a stretch of scale factor k, to obtain $y = f(kx)$ from $y = f(x)$, where $k > 1$ (though with equality when $x = 1$, rather than

$x = 0$).]



$y = \ln(x-1)$ & $y = \ln(x^2-1)$

(iii) $y = ln|x^2 - 1|$

For $|x| > 1$, $ln|x^2 - 1| = \ln(x^2 - 1)$

For $x = 1$, $ln|x^2 - 1|$ is undefined

For $|x| < 1$, $ln|x^2 - 1| = \ln(1 - x^2)$

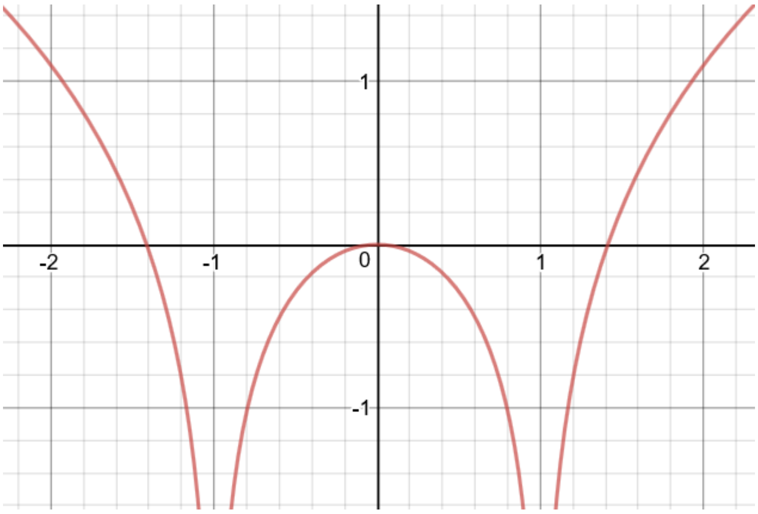
$y = ln(1 - x^2)$ is an even function; ie it's symmetric about the

y-axis. We need therefore only consider the curve for $0 \leq x < 1$.

For $0 \le x < 1$, $y = ln(1 - x^2)$ will be similar to $y = ln(1 - x)$.

For $x = \frac{1}{2}$, for example, the y-coordinate will be $ln(1 - \frac{1}{4})$; ie we are looking to the left (to obtain $x = \frac{1}{4}$), and dragging the graph of $y = ln(1 - x)$ back to the right. Thus $y = ln(1 - x^2)$ hugs the line $x = 1$ (and also $y = 0$) more than $y = ln(1 - x)$.

[Compare with the graphs $y = x^2$ and $y = x^4$, where the latter is 'squarer'.]



$y = ln|x^2 - 1|$