STEP Exercises - Turning points, Points of inflexion & Cubics - Sol'ns (6 pages; 17/9/18)

(1) What can be said about the graph of $f(x)$ if $(x-a)^n$ is a factor of $f(x)$, where $f(x)$ is a polynomial function and $n \in \mathbb{Z}^+$?

Solution

There is a turning point at $x = a$ if n is even, and a point of inflexion if

$n > 1$ is odd.

(2) Find the turning points of $y = (x^2 - 4x + 3)^2$

Solution

Method 1

As $x^2 - 4x + 3 = (x-1)(x-3)$,

$y = (x-1)^2(x-3)^2$

Then $\frac{dy}{dx} = 2(x-1)(x-3)^2 + (x-1)^2(2)(x-3)$

$= 2(x-1)(x-3)(x-3+x-1)$

$= 4(x-1)(x-3)(x-2)$

$\frac{dy}{dx} = 0$ when $x = 1, 2$ & 3

At $x = 1$, $\frac{dy}{dx}$ changes from –ve to +ve, indicating a min. point.

At $x = 2$, $\frac{dy}{dx}$ changes from +ve to -ve, indicating a max. point.

At $x = 3$, $\frac{dy}{dx}$ changes from –ve to +ve, indicating a min. point.

The min. points are therefore (1, 0) and (3, 0), whilst the max. is at (2,1).

Method 2

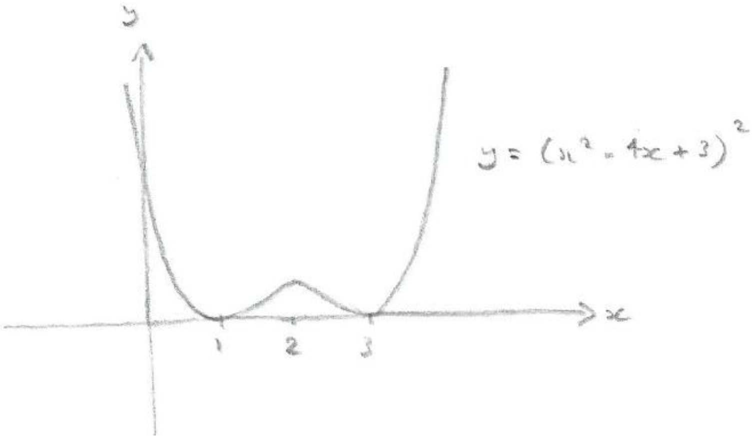
$(x^2 - 4x + 3)^2 \geq 0$ and $(x^2 - 4x + 3)^2 = (x-1)^2(x-3)^2 = 0$ has roots at $x = 1$ & 3, so that there are minima at these two points.

For $x = 1 - t$, $y = x^2 - 4x + 3$ and hence $y = (x^2 - 4x + 3)^2$ increases as t increases, and similarly for $x = 3 + t$.

For $1 < x < 3$, $y = (x^2 - 4x + 3)^2$ attains a max. when

$x^2 - 4x + 3$ (which is negative in this range) is at a min. ; ie when

$x = 2$.



(3) (i) For $f(x) = ax^3 + bx^2 + cx + d$, what is the x-coordinate of the point of inflexion?

(ii) Give examples of cubic functions for which the PoI is at the Origin, and the gradient at the Origin is (a) 1 (b) -1. How do the shapes of the two graphs differ?

Solution

(i) $f'(x) = 3ax^2 + 2bx + c$; $f''(x) = 6ax + 2b$

$f''(x) = 0 \Rightarrow x = -\frac{b}{3a}$

(ii) As the graph passes through the Origin, we can consider

$f(x) = ax^3 + bx^2 + cx$

From (2), PoI is at $x = -\frac{b}{3a}$, so that $b = 0$

$f'(x) = 3ax^2 + c$

$f'(0) = 1 \Rightarrow c = 1$

eg $y = 2x^3 + x = x(2x^2 + 1)$

$f'(0) = -1 \Rightarrow c = -1$

eg $y = 2x^3 - x = x(2x^2 - 1)$

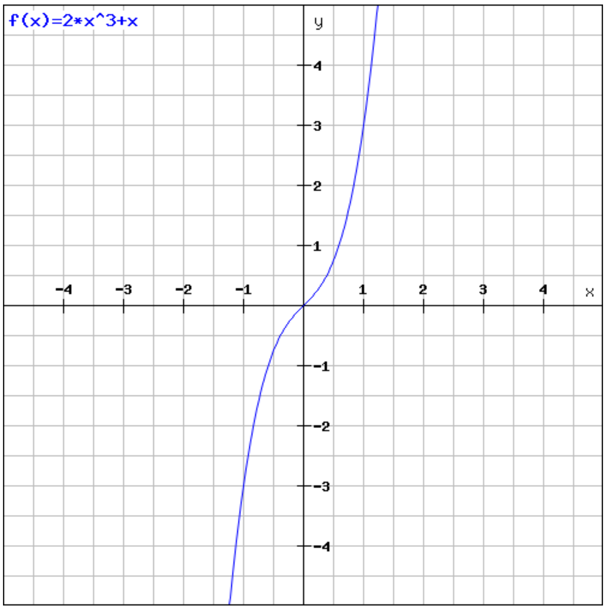


Fig. 2: $y = 2x^3 + x$

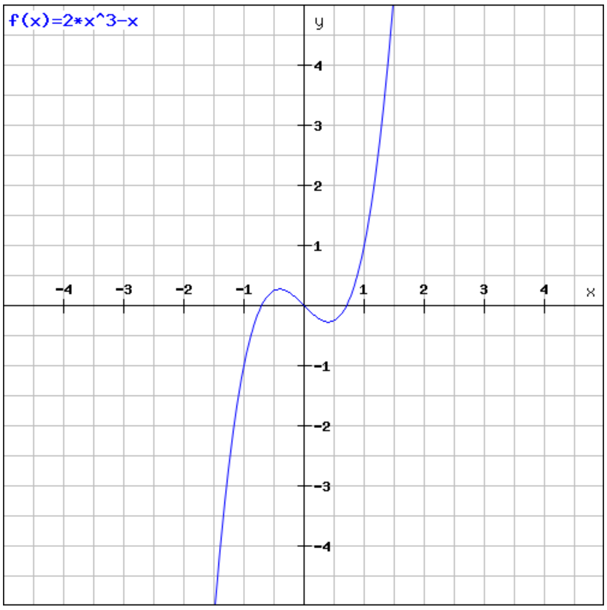
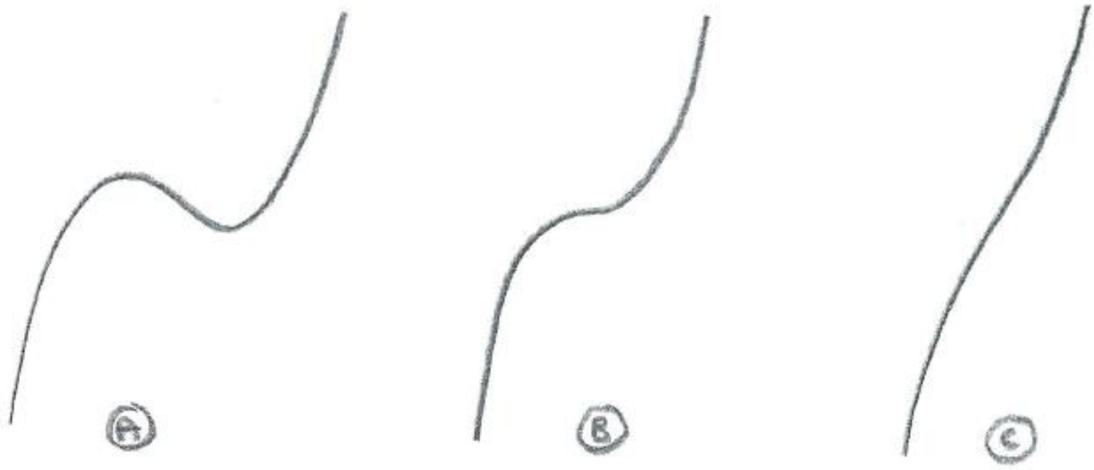


Fig. 3: $y = 2x^3 - x$

(4) Cubics

(i) What possible shapes might a cubic have (ignoring its position relative to the axes)?



A, B & C have 2, 1 & 0 stationary points respectively. These are for cases where the coefficient of x^3 is positive; so inverted shapes are also possible.

(ii) How many stationary points does the cubic function,

$f(x) = x^3 + x^2 - 2x + 3$ have?

$f'(x) = 3x^2 + 2x - 2$

To find the number of solutions to $f'(x) = 0$,

consider the discriminant: $2^2 - 4(3)(-2) > 0$

Thus there are 2 stationary points.

(iii) What is the condition for there to be 2 stationary points for the general cubic $f(x) = ax^3 + bx^2 + cx + d$?

2 sol′ns of $f'(x) = 3ax^2 + 2bx + c = 0$

$\Rightarrow (2b)^2 - 4(3a)c > 0$

$\Rightarrow b^2 - 3ac > 0$

(iv) For $f(x) = ax^3 + bx^2 + cx + d$, find the x-coordinate of any turning points of the gradient.

For a stationary point of the gradient, we want $\frac{d}{dx}(f'(x)) = 0$; ie $f''(x) = 0$:

$f'(x) = 3ax^2 + 2bx + c$

$f''(x) = 6ax + 2b$

$f''(x) = 0 \Rightarrow x = -\frac{b}{3a}$

And $\frac{d^2}{dx^2}\left(f'(x)\right) = f'''(x) = 6a > 0$, so that the stationary point is a minimum (ie it is a turning point).

A turning point of the gradient is the definition of a point of inflexion (or inflection).

Thus, all cubics have one point of inflexion. They can be shown to have rotational symmetry about this point.

If the cubic has turning points, how could they be used to find the point of inflexion?

By symmetry, the coordinates of the point of inflexion will be halfway between those of the turning points.

(v) For $f(x) = ax^3 + bx^2 + cx + d$, find conditions for the shape of the curve to be each of the 3 possibilities shown in (i), by considering the gradient at the point of inflexion.

$f'\left(-\frac{b}{3a}\right) = \frac{b^2}{3a} - \frac{2b^2}{3a} + c = c - \frac{b^2}{3a}$

Diagram (A): Either (i) $a > 0$ & $f'\left(-\frac{b}{3a}\right) < 0$

or (ii) $a < 0$ & $f'\left(-\frac{b}{3a}\right) > 0$

(i): $3\text{ac} - b^2 < 0 \Leftrightarrow b^2 - 3ac > 0$ [agreeing with part (iii)]

(ii): $3\text{ac} - b^2 < 0$ also

Diagram (B): Stationary point of inflexion $\Leftrightarrow$ $f'\left(-\frac{b}{3a}\right) = 0$

$\Leftrightarrow b^2 - 3ac = 0$

Diagram (C): Either (i) $a > 0$ & $f'\left(-\frac{b}{3a}\right) > 0$

or (ii) $a < 0$ & $f'\left(-\frac{b}{3a}\right) < 0$,

so that $b^2 - 3ac < 0$