STEP Exercises - Quadratics & Polynomials (Sol'ns)

(4 pages; 9/9/18)

(1) For what value of x does $(x+2)(x+4)$ have its minimum value?

Solution

Roots of $(x+2)(x+4)=0$ are $-2\ \&-4$, so minimum is at

$x=-3$ (or complete the square, or find stationary point)

(2) Consider the quadratic equation $x^2+bx+c=0$

(i) By experimenting with different examples, find conditions on $b\ and/or\ c$ for the roots of the equation to exist and be of the same sign.

(ii) Find conditions for the roots to exist and both be positive

Solution

(i) First of all, $b^2-4c\geq 0$ is needed, in order for there to be real roots.

Then $c>0$ will ensure that the roots are either both positive or both negative; whilst $c<0$ ensures that the roots are of different sign (see Fig. 1 below). Clearly $c=0$ gives one zero root.

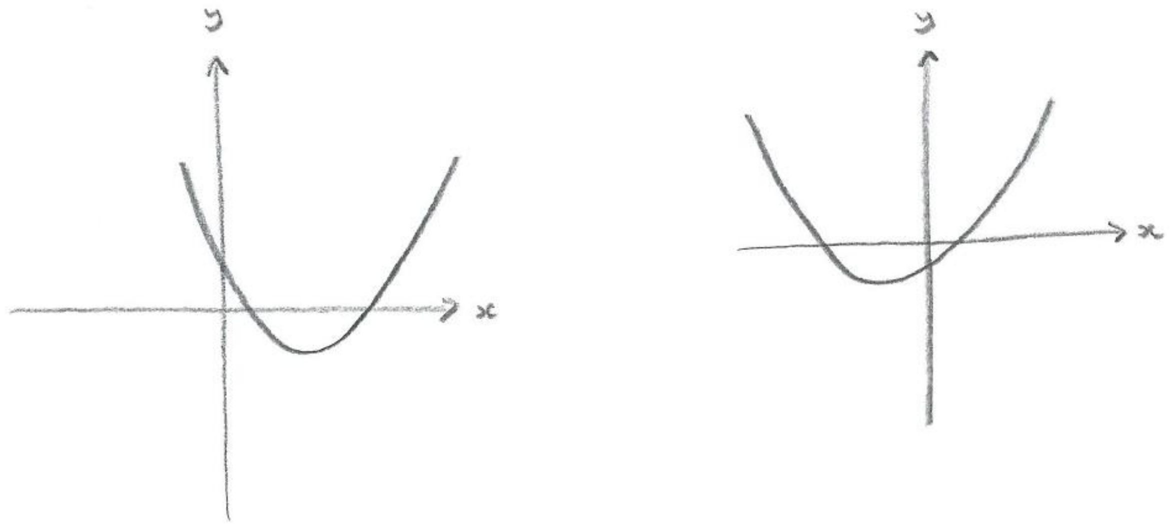

Fig. 1

So we want $b^2 \geq 4c$ & $c > 0$

or $c > 0$ & $|b| \geq 2\sqrt{c}$ (as an alternative form)

(ii) In order for the roots both to be positive, we require the above conditions to apply, and in addition for the minimum to have a positive x coordinate.

As the minimum is halfway between the two roots (assuming they exist), and in this example has x coordinate $-\frac{b}{2}$ (from the quadratic formula; in general it is $-\frac{b}{2a}$), we therefore want $b < 0$.

So our conditions become:

$c > 0$ & $|b| \geq 2\sqrt{c}$ & $b < 0$

which simplifies to

$c > 0$ & $b < -2\sqrt{c}$

Alternative Approach

We could instead require the smaller of the two roots to be positive, so that $-b - \sqrt{b^2 - 4c} > 0$

$\Leftrightarrow -b > \sqrt{b^2 - 4c}$

$\Leftrightarrow b < 0 \ \& \ |b| > \sqrt{b^2 - 4c}$

$\Leftrightarrow b < 0$ & $c > 0$, together with $b^2 - 4c \geq 0$ (in order for the roots to exist), as before

(3) (i) Factorise (a) $x^3 - y^3$ (b) $x^3 + y^3$

(ii) Can $3^{54} - 2^{54}$ be prime?

Solution

(i)(a) Let $f(x) = x^3 - y^3$

By the Factor theorem (treating $f(x)$ as a cubic in x), since

$f(y) = 0$, $(x - y)$ is a factor of $x^3 - y^3$, leading to

$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

(b) Similarly, $(x + y)$ is a factor of $x^3 + y^3$, leading to

$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

More generally,

$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$

and, if n is odd:

$x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + \cdots - xy^{n-2} + y^{n-1})$

(Note the alternating signs in the 2nd bracket; consider for example $x = y = 1$.)

[Note that $x^n + y^n \geq 0$ when n is even, and $x^n + y^n = 0$ only when $x = y = 0$ (ie not for all x & y); and so there are no linear factors.]

(ii) We could consider using the result

$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$

but it isn't of any use having $x - y = 3 - 2 = 1$.

However, we can write $3^{54} - 2^{54}$ as $(3^{18})^3 - (2^{18})^3$, for example, to give the factor $3^{18} - 2^{18}$ (similarly, $3^3 - 2^3$ is also a factor).

[Alternatively, we could just write $3^{54} - 2^{54}$ as $(3^{27})^2 - (2^{27})^2$, and use the difference of two squares.]

So $3^{54} - 2^{54}$ isn't a prime number.

(4) Factorise $2x^3 - 33x^2 - 6x + 11$

Solution

If the factorisation is of the form $= (x + a)(2x^2 + bx + c)$,

then a must be $\pm$ a factor of 11

Applying the factor theorem, this is found not to be the case.

Let $2x^3 - 33x^2 - 6x + 11 = (2x + a)(x^2 + bx + c)$,

Equating coefficients gives:

$-33 = 2b + a, -6 = 2c + ab$ & $11 = ac$

Testing the possible combinations of a & c ($\pm$ the factors of 11) shows that $a = -1, c = -11$ & $b = -16$

ie $2x^3 - 33x^2 - 6x + 11 = (2x - 1)(x^2 - 16x - 11)$.