

STEP Exercises - Inequalities (sol'ns) (6 pages; 20/9/18)

(1) Are the following true or false?

(i) $a < b \Rightarrow \frac{1}{a} > \frac{1}{b}$

(ii) $a < b \Rightarrow a^2 < b^2$

(iii) $a < b \ \& \ c < d \Rightarrow a + c < b + d$

(iv) $a < b \ \& \ c < d \Rightarrow a - c < b - d$

Solution

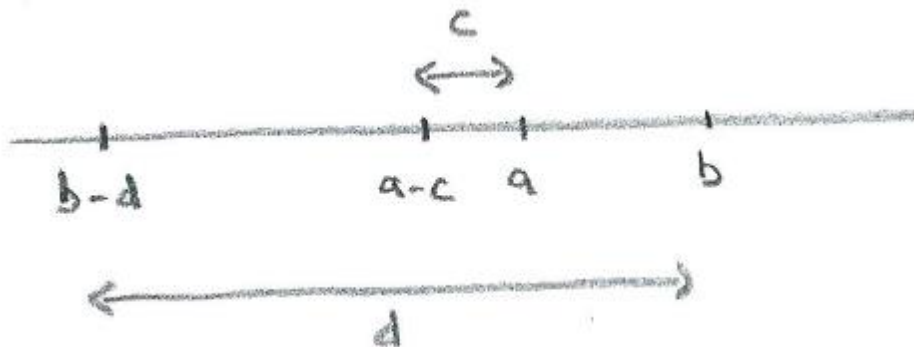
(i) Not true if $a < 0 \ \& \ b > 0$ (consider the graph of $y = 1/x$)

(ii) Not true if $a < 0 \ \& \ b < 0$ or

if $a < 0, b > 0 \ \& \ |b| < |a|$ (consider the graph of $y = x^2$)

(iii) True: $a < b \Rightarrow a + c < b + c < b + d$

(iv) False: For example, $8 < 9$ and $2 < 4$, but it is not true that $8 - 2 < 9 - 4$; see diagram



(2) Prove or provide a counter-example for the conjecture

$x > a$ & $y > b \Rightarrow xy > ab$ (a, b real) in each of the following cases:

(i) $a > 0, b > 0$ (ii) $a < 0, b < 0$ (iii) $a > 0, b < 0$

Solution

(i) $x > a \Rightarrow xy > ay$ [as $y > 0$] $> ab$ [since $y > b \Rightarrow ay > ab$]

so true

[or refer to graph of $y = ab$]

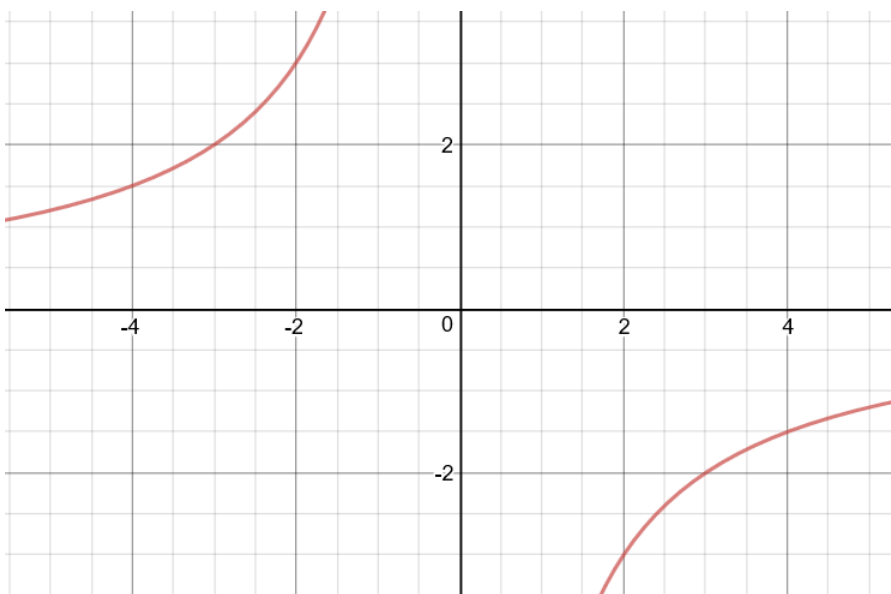
(b) $a < 0, b < 0$

counter-example: $x = 0$

(c) $a > 0, b < 0$

consider graph of $xy = ab$ when $a = 3, b = -2$ (see below)

(counter-example: $x = 4 + \delta, y = -2 + \delta$)



(3) Prove that $a + b < 1 + ab$ if $a > 1$ and $b > 1$

Solution

$$\Leftrightarrow a + b - 1 - ab < 0$$

$$\Leftrightarrow a(1 - b) - (1 - b) < 0$$

$$\Leftrightarrow (a - 1)(1 - b) < 0$$

(4) Prove that $\frac{a}{b} < \frac{a+c}{b+c}$ where $a, b, c > 0 \Leftrightarrow a < b$

Proof

$$\frac{a}{b} < \frac{a+c}{b+c} \Leftrightarrow a(b+c) < b(a+c)$$

$$\Leftrightarrow ac < bc \Leftrightarrow a < b$$

(5) Let x, y & z be positive real numbers.

(i) If $x + y \geq 2$, is it necessarily true that $\frac{1}{x} + \frac{1}{y} \leq 2$?

(ii) If $x + y \leq 2$, is it necessarily true that $\frac{1}{x} + \frac{1}{y} \geq 2$?

Solution

(i) No: if x (say) is very small, then $\frac{1}{x}$ will be very large.

(ii) Note that, when $x = y = 1$, $\frac{1}{x} + \frac{1}{y} = 2$

Also, if the result is true for $x + y = 2$, then if x or y is made smaller, so that $x + y < 2$, $\frac{1}{x} + \frac{1}{y}$ becomes larger, so that the result is still true. So, WLOG (without loss of generality), we need only investigate the case where $x + y = 2$.

[This is an example of "reformulating the problem".]

Experimenting with some numbers, we get the impression that $\frac{1}{x} + \frac{1}{y} \geq 2$. So, aiming for a proof by contradiction, suppose that $\frac{1}{x} + \frac{1}{y} < 2$

Then, $\frac{x+y}{xy} < 2$, so that $2 < 2x(2 - x)$ [as $xy > 0$]

and hence $1 < 2x - x^2$ and $x^2 - 2x + 1 < 0$ or $(x - 1)^2 < 0$, which is impossible.

Thus $\frac{1}{x} + \frac{1}{y} \geq 2$ when $x + y \leq 2$

Alternative approach

To prove that $\frac{1}{x} + \frac{1}{y} \geq 2$ when $x + y = 2$,

we note that WLOG we need only consider solutions of the form $x = 1 + \delta, y = 1 - \delta$ (where $\delta > 0$).

But the reduction from $\frac{1}{1}$ to $\frac{1}{1+\delta}$ will be outweighed by the rise from $\frac{1}{1}$ to $\frac{1}{1-\delta}$ [consider the extreme cases $\frac{1}{1000}$ to $\frac{1}{1001}$ versus $\frac{1}{4}$ to $\frac{1}{3}$, which shows that the change of 1 in the denominator has a greater effect when the denominator is smaller, as it is with $1 - \delta$, compared to $1 + \delta$]

(6) Assuming that $\sin^2\theta + \cos^2\theta = 1$, but without using any compound angle results, show that $\sin\theta\cos\theta \leq \frac{1}{2}$

Solution

$$(\sin\theta - \cos\theta)^2 \geq 0 \Rightarrow \sin^2\theta + \cos^2\theta - 2\sin\theta\cos\theta \geq 0$$

$$\Rightarrow 1 \geq 2\sin\theta\cos\theta \Rightarrow \sin\theta\cos\theta \leq \frac{1}{2}$$

(7) Which is larger: $\frac{\sqrt{7}}{2}$ or $\frac{1+\sqrt{6}}{3}$ (without using a calculator)?

Solution

Considering the difference of squares:

$$\frac{7}{4} - \frac{(1+2\sqrt{6}+6)}{9} = \frac{63-28-8\sqrt{6}}{36} > \frac{35-8(3)}{36} > 0 ; \text{ so } \frac{\sqrt{7}}{2} \text{ is larger}$$

[Another approach is to investigate $\frac{\left(\frac{7}{4}\right)}{\left(\frac{7+2\sqrt{6}}{9}\right)} = \frac{63(7-2\sqrt{6})}{4(49-24)} =$

$\frac{63(7-2\sqrt{6})}{100}$, but it isn't as easy to show that this expression is greater than 1]

(8) Is $\frac{6}{7} < \frac{2}{\sqrt{5}}$?

Solution

$$\frac{6}{7} < \frac{2}{\sqrt{5}} \Leftrightarrow \frac{36}{49} < \frac{4}{5}$$

$$49 \times 0.8 = \frac{1}{10} (320 + 72) = 39.2 > 36$$

$$\text{So } \frac{36}{49} < \frac{39.2}{49} = 0.8 = \frac{4}{5}$$

Answer is Yes.

(9) Show that $e^3 > 4e^{\frac{3}{2}}$

Solution

An equivalent result to prove is $e^{\frac{3}{2}} > 4$ (dividing both sides by $e^{\frac{3}{2}}$, which is positive) [you can never be sure what counts as being obvious]

$\Leftrightarrow e^3 > 16$ (as the function $y = x^2$ is increasing for $x > 0$)

$$e^3 > (2 + 0.7)^3 > 2^3 + 3(2^2)(0.7) = 8 + 8.4 > 16,$$

so that the original result is also true

(10) Is $\log_2 3 > \frac{3}{2}$?

Solution

$$\log_2 3 > \frac{3}{2} \Leftrightarrow 3 > 2^{\frac{3}{2}} \text{ (as } y = 2^x \text{ is an increasing function)}$$

$$\Leftrightarrow 3^2 > 2^3$$

So answer is Yes.