STEP Exercises - Inequalities (sol'ns) (6 pages; 6/10/19)

(1) Are the following true or false?

(i) $a < b \Rightarrow \frac{1}{a} > \frac{1}{b}$

(ii) $a < b \Rightarrow a^2 < b^2$

(iii) $a < b \;\&\; c < d \Rightarrow a + c < b + d$

(iv) $a < b \;\&\; c < d \Rightarrow a - c < b - d$

Solution

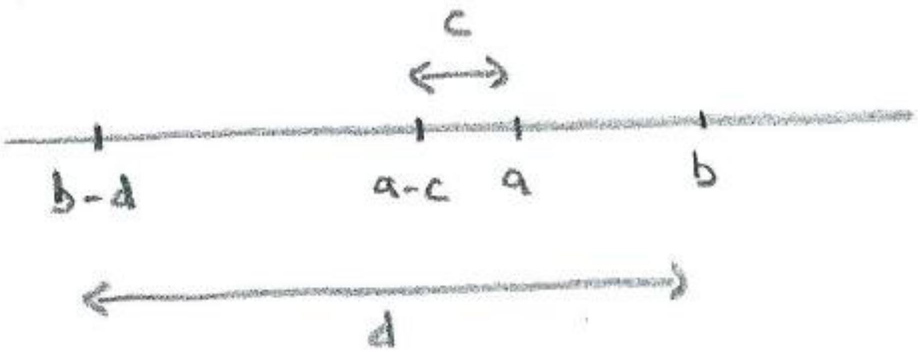
(i) Not true if $a < 0 \;\&\; b > 0$ (consider the graph of $y = 1/x$)

(ii) Not true if $a < 0 \;\&\; b < 0$ or

if $a < 0, b > 0 \;\&\; |b| < |a|$(consider the graph of $y = x^2$)

(iii) True: $a < b \Rightarrow a + c < b + c < b + d$

(iv) False: For example, $8 < 9$ and $2 < 4$, but it is not true that $8 - 2 < 9 - 4$; see diagram



(2) Prove or provide a counter-example for the conjecture

$x > a \,\&\, y > b \Rightarrow xy > ab$ $(a, b\ real)$ in each of the following cases:

(i) $a > 0, b > 0$ (ii) $a < 0, b < 0$ (iii) $a > 0, b < 0$

Solution

(i) $x > a \Rightarrow xy > ay$ [as $y > 0$] $> ab$ [since $y > b \Rightarrow ay > ab$]

so true

[or refer to graph of $y = ab$]

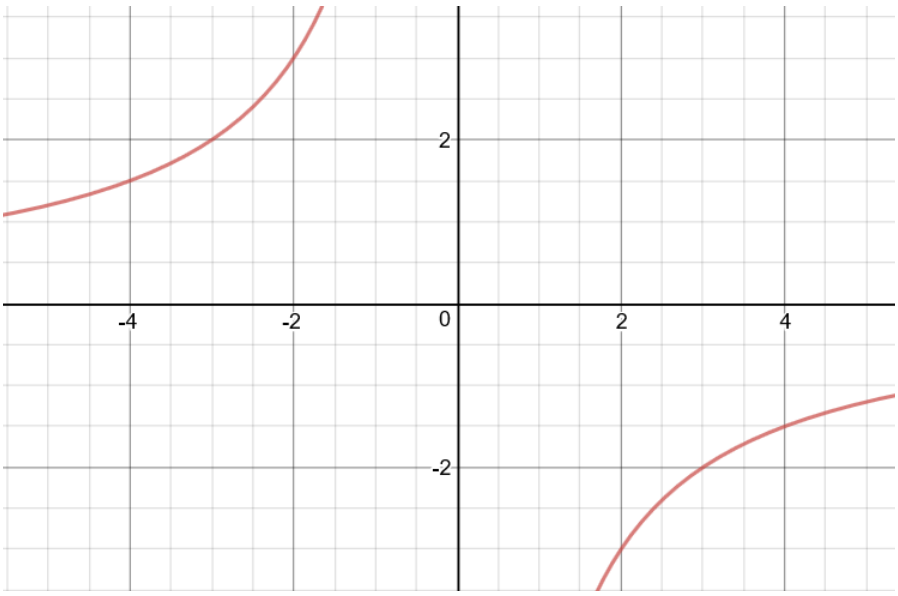
(b) $a < 0, b < 0$

counter-example: $x = 0$

(c) $a > 0, b < 0$

consider graph of $xy = ab$ when $a = 3, b = -2$ (see below)

(counter-example: $x = 4 + \delta, y = -2 + \delta$)



(3) Prove that $a + b < 1 + ab$ if $a > 1$ and $b > 1$

Solution

$\Leftrightarrow a + b - 1 - ab < 0$

$\Leftrightarrow a(1 - b) - (1 - b) < 0$

$\Leftrightarrow (a - 1)(1 - b) < 0$

(4) Prove that $\frac{a}{b} < \frac{a+c}{b+c}$ where $a, b, c > 0 \Leftrightarrow a < b$

Proof

$\frac{a}{b} < \frac{a+c}{b+c} \Leftrightarrow a(b + c) < b(a + c)$

$\Leftrightarrow ac < bc \Leftrightarrow a < b$

(5) Let x, y & z be positive real numbers.

(i) If $x + y \geq 2$, is it necessarily true that $\frac{1}{x} + \frac{1}{y} \leq 2$?

(ii) If $x + y \leq 2$, is it necessarily true that $\frac{1}{x} + \frac{1}{y} \geq 2$?

Solution

(i) No: if x (say) is very small, then $\frac{1}{x}$ will be very large.

(ii) Note that, when $x = y = 1, \frac{1}{x} + \frac{1}{y} = 2$

Also, if the result is true for $x + y = 2$, then if x or y is made smaller, so that $x + y < 2, \frac{1}{x} + \frac{1}{y}$ becomes larger, so that the result is still true. So, WLOG (without loss of generality), we need only investigate the case where $x + y = 2$.

[This is an example of "reformulating the problem".]

Experimenting with some numbers, we get the impression that $\frac{1}{x}+\frac{1}{y}\geq 2$. So, aiming for a proof by contradiction, suppose that $\frac{1}{x}+\frac{1}{y}<2$

Then, $\frac{x+y}{xy}<2$, so that $2<2x(2-x)$ [as $xy>0$]

and hence $1<2x-x^2$ and $x^2-2x+1<0$ *or* $(x-1)^2<0$,

which is impossible.

Thus $\frac{1}{x}+\frac{1}{y}\geq 2$ when $x+y\leq 2$

Alternative approach

To prove that $\frac{1}{x}+\frac{1}{y}\geq 2$ when $x+y=2$,

we note that WLOG we need only consider solutions of the form $x=1+\delta, y=1-\delta$ (where $\delta>0$).

But the reduction from $\frac{1}{1}$ to $\frac{1}{1+\delta}$ will be outweighed by the rise from $\frac{1}{1}$ to $\frac{1}{1-\delta}$ [consider the extreme cases $\frac{1}{1000}$ *to* $\frac{1}{1001}$ versus $\frac{1}{4}$ *to* $\frac{1}{3}$, which shows that the change of 1 in the denominator has a greater effect when the denominator is smaller, as it is with $1-\delta$, compared to $1+\delta$]

(6) Assuming that $sin^2\theta+cos^2\theta=1$, but without using any compound angle results, show that $sin\theta cos\theta\leq\frac{1}{2}$

Solution

$(sin\theta-cos\theta)^2\geq 0\Rightarrow sin^2\theta+cos^2\theta-2sin\theta cos\theta\geq 0$

$\Rightarrow 1\geq 2sin\theta cos\theta\Rightarrow sin\theta cos\theta\leq\frac{1}{2}$

(7) Which is larger: $\frac{\sqrt{7}}{2}$ or $\frac{1+\sqrt{6}}{3}$ (without using a calculator)?

Solution

Considering the difference of squares:

$\frac{7}{4} - \frac{(1+2\sqrt{6}+6)}{9} = \frac{63-28-8\sqrt{6}}{36} > \frac{35-8(3)}{36} > 0$; so $\frac{\sqrt{7}}{2}$ is larger

[Another approach is to investigate $\frac{\left(\frac{7}{4}\right)}{\left(\frac{7+2\sqrt{6}}{9}\right)} = \frac{63(7-2\sqrt{6})}{4(49-24)} =$ $\frac{63(7-2\sqrt{6})}{100}$, but it isn't as easy to show that this expression is greater than 1]

(8) Is $\frac{6}{7} < \frac{2}{\sqrt{5}}$?

Solution

$\frac{6}{7} < \frac{2}{\sqrt{5}} \Leftrightarrow \frac{36}{49} < \frac{4}{5}$

$49 \times 0.8 = \frac{1}{10}(320 + 72) = 39.2 > 36$

So $\frac{36}{49} < \frac{39.2}{49} = 0.8 = \frac{4}{5}$

Answer is Yes.

(9) Show that $e^3 > 4e^{\frac{3}{2}}$

Solution

An equivalent result to prove is $e^{\frac{3}{2}} > 4$ (dividing both sides by $e^{\frac{3}{2}}$, which is positive) [you can never be sure what counts as being obvious]

$\Leftrightarrow e^3 > 16$ (as the function $y = x^2$ is increasing for $x > 0$)

$e^3 > (2 + 0.7)^3 > 2^3 + 3(2^2)(0.7) = 8 + 8.4 > 16$,

so that the original result is also true

(10) Is $log_2 3 > \frac{3}{2}$?

Solution

$log_2 3 > \frac{3}{2} \Leftrightarrow 3 > 2^{\frac{3}{2}}$ (as $y = 2^x$ is an increasing function)

$\Leftrightarrow 3^2 > 2^3$

So answer is Yes.

(11) Use differentiation to show that $lnx \geq 1 - \frac{1}{x}$ for $x > 0$

Solution

$\frac{d}{dx}(lnx) = \frac{1}{x}$ and $\frac{d}{dx}\left(1 - \frac{1}{x}\right) = \frac{1}{x^2}$

For $0 < x < 1, \frac{1}{x} < \frac{1}{x^2}$; ie $1 - \frac{1}{x}$ is increasing faster than lnx

For $x > 1, \frac{1}{x} > \frac{1}{x^2}$; ie lnx is increasing faster than $1 - \frac{1}{x}$

[as can be seen from a sketch of the two curves]

When $x = 1$, $lnx = 0$ and $1 - \frac{1}{x} = 0$.

Thus, for $0 < x < 1$, $1 - \frac{1}{x}$ is catching up with lnx, and for $x > 1$, lnx moves away from $1 - \frac{1}{x}$, and hence $lnx \geq 1 - \frac{1}{x}$ for $x > 0$.