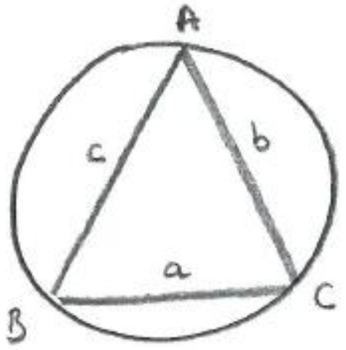
STEP Exercises - Geometry (sol'ns) (6 pages; 30/9/18)

(1) ABC is a triangle circumscribed by a circle of radius R, as shown in the diagram below.

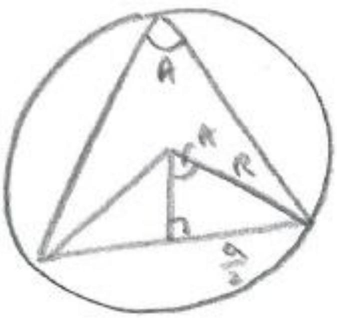


Show that (i) $\frac{a}{sinA} = 2R$ (ii) the area of the triangle is $\frac{abc}{4R}$

Solution

(i) Drawing radii from B and C to the centre of the circle, as in the diagram below, and noting that the angle at the centre is twice the angle at the circumference,

$\sin A = \frac{\left(\frac{a}{2}\right)}{R}$, so that $\frac{a}{sinA} = 2R$, as required



(ii) Area of $ABC = \frac{1}{2}bcsinA = \frac{1}{2}bc\left(\frac{a}{2R}\right) = \frac{abc}{4R}$

(2) Find the equation of the circle passing through the points

A (2,8) , B (7,3) and D (5,7)

Solution

The first step is to find the centre of the circle, using the fact that the perpendicular bisector of each chord passes through the centre.

The chord AB has mid-point (9/2, 11/2)

and gradient $\frac{3-8}{7-2} = -1$

The perpendicular bisector of AB therefore has equation

$\frac{y-11/2}{x-9/2} = -\frac{1}{-1}$

$\rightarrow 2y - 11 = 2x - 9$

$\rightarrow y = x + 1$

The chord BD has mid-point (6, 5)

and gradient $\frac{7-3}{5-7} = -2$

The perpendicular bisector of BD therefore has equation

$\frac{y-5}{x-6} = -\frac{1}{-2}$

$\rightarrow y = \frac{1}{2}x + 2$

The centre of the circle C is then found from the intersection of these lines:

$x + 1 = \frac{1}{2}x + 2$

so that $x = 2$ and $y = 3$

The radius is then the distance CA (for example)

$= \sqrt{(2-2)^2 + (3-8)^2} \ = 5$

Hence the equation of the circle is $(x-2)^2 + (y-3)^2 = 25$

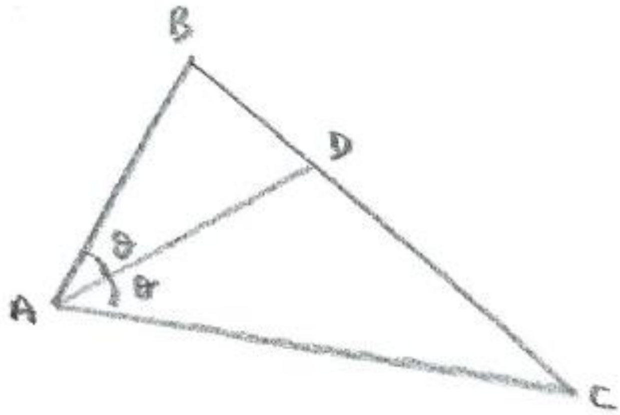
(Check: B and D satisfy the equation.)

(3) Angle Bisector Theorem

Referring to the diagram below, the Angle Bisector theorem says that

$\frac{BD}{DC} = \frac{AB}{AC}$

Prove the Angle Bisector Theorem.



Solution

By the Sine rule for triangle ABD, $\frac{BD}{sin\theta} = \frac{AB}{sinADB}$ (1)

and, for triangle ADC, $\frac{DC}{sin\theta} = \frac{AC}{sinADC} = \frac{AC}{sinADB}$ (2)

Then (1) $\Rightarrow \frac{sin\theta}{sinADB} = \frac{BD}{AB}$ and (2) $\Rightarrow \frac{sin\theta}{sinADB} = \frac{DC}{AC}$

so that $\frac{BD}{AB} = \frac{DC}{AC}$

and hence $\frac{BD}{DC} = \frac{AB}{AC}$

(4) Find the shortest and longest distances from the point (3,4) to the circle $x^2 + y^2 = 144$ (justifying any assumptions you make).

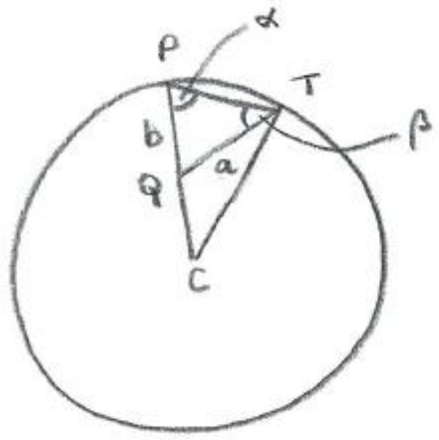
Solution

Approach 1

The nearest point P_1 on the circle to Q (3,4) will be such that P_1Q is perpendicular to the tangent at P_1.

This can be justified as follows:

If PQ is perpendicular to the tangent at P, then PQ extended will pass through the centre of the circle, C (by a property of the tangent to a circle). In the diagram below, we wish to show that $a > b$ (where $a = QT$ & $b = QP$); ie that any neighbouring point, T on the circle will be further from Q than P.



[Note: This diagram and the following one are applicable for any point Q.]

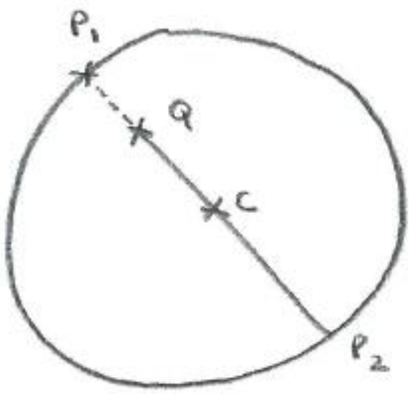
As CPT is an isoceles triangle, angles CPT and CTP are equal, and hence $\alpha > \beta$. Then $\frac{a}{sin\alpha} = \frac{b}{sin\beta}$ and $\frac{a}{b} = \frac{sin\alpha}{sin\beta} > 1$ (as $\alpha > \beta$, and $sinx$ is an increasing function for $0 < x < \frac{\pi}{2}$), so that $a > b$, as required.

Thus we have shown that P_1Q is perpendicular to the tangent at P_1, and hence (as already mentioned) Q lies on the radius from P_1.

So the distance $P_1Q = 12 - 5 = 7$, since $OQ = 5$ (by Pythagoras).

To find the furthest point from Q:

Let P be a point on the circle and P_2 the point furthest from Q. The maximum value that PQ can possibly take is the diameter less P_1Q, and this can be achieved if P_1, Q, C & P_2 lie on the same diameter, as in the diagram below.



In this situation, $P_2Q = P_2C + CQ = 12 + 5 = 17$

Approach 2

Let (x, y) be a point P on the circle, so that $x^2 + y^2 = 144$ (A)

The square of the distance from (3,4) to P is:

$D^2 = (x - 3)^2 + (y - 4)^2$

We wish to find stationary points of D^2:

$\frac{d}{dx}D^2 = 0 \ \Rightarrow 2(x-3) + 2(y-4)\frac{dy}{dx} = 0$ (B)

Then, differentiating (A), $2x + 2y.\frac{dy}{dx} = 0$ (C)

and (B) & (C) give $(x-3) + (y-4)\left(-\frac{x}{y}\right) = 0$,

so that $xy - 3y - xy + 4x = 0$,

and hence $y = \frac{4x}{3}$

This is the equation of the diameter through $(4,3)$, which establishes that P_1 , P_2 & Q all lie on a diameter. The lengths P_1Q and P_2Q are then calculated as before.