STEP Exercises: Curve sketching (sol'ns)

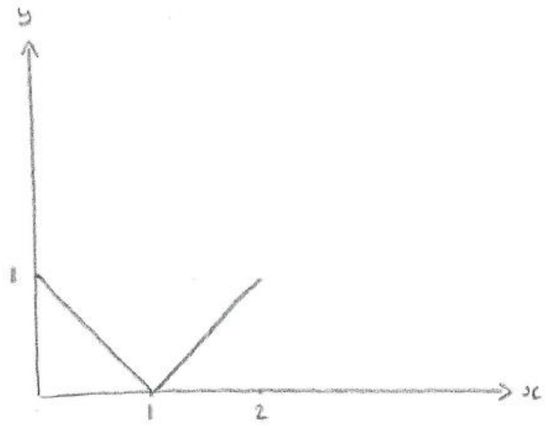
(5 pages; 19/9/18)

(1) Sketch the graph of $\sqrt{x^2-2x+1}$ for $0 \le x \le 2$

Solution: (see sketch below)

For $0 \le x \le 1$, $\sqrt{x^2-2x+1} = \sqrt{(x-1)^2} = \sqrt{(1-x)^2} = 1-x$

For $1 \le x \le 2$, $\sqrt{x^2-2x+1} = \sqrt{(x-1)^2} = x-1$



(2) Sketch the curve $x^2 = y(1-y)$

[Hint: Rearrange]

Solution

$y(1-y) = -(y^2-y) = -\left(y-\frac{1}{2}\right)^2 + \frac{1}{4}$

So curve is $x^2 + \left(y-\frac{1}{2}\right)^2 = \frac{1}{4}$

(3) Sketch $x^n \pm y^n = 1$ for large n

Solution

Consider odd and even n separately.

(i) $x^n - y^n = 1$; odd n

For large positive x & y, $y^n = x^n - 1$, so that $y \approx x$; ie $y = x$ is an asymptote. Also for large negative x & y.

Negative x with positive y is not possible.

Positive x with negative y is only possible if both $|x|$ & $|y|$ are < 1.

Curve passes through $(0, -1)$ and $(1, 0)$.

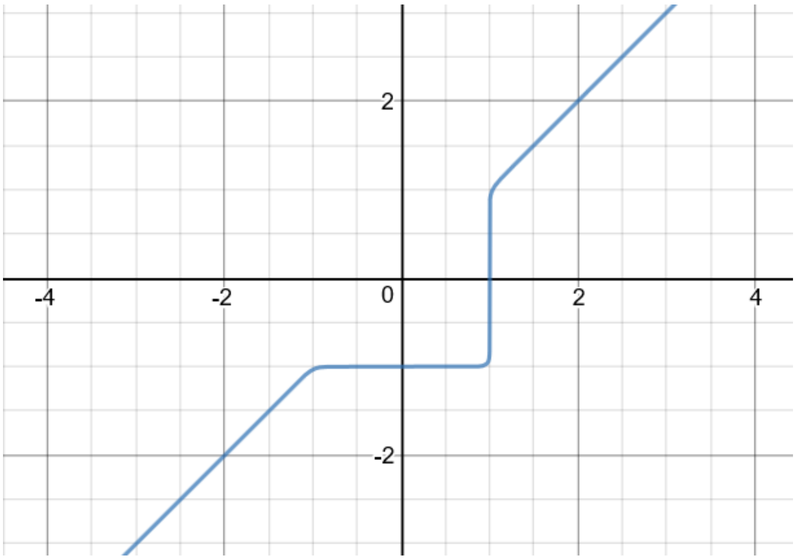
When $|x| < 1, x^n$ can be made as small as we like, by making n large enough. Then $x^n - y^n = 1 \Rightarrow y \to -1$ as $n \to \infty$.

Similarly, for $|y| < 1, x \to 1$ as $n \to \infty$.

For $x > 1, x^n$ can be made as large as we like, by making n large enough. Then $x^n - y^n = 1 \Rightarrow y \to x$ as $n \to \infty$.

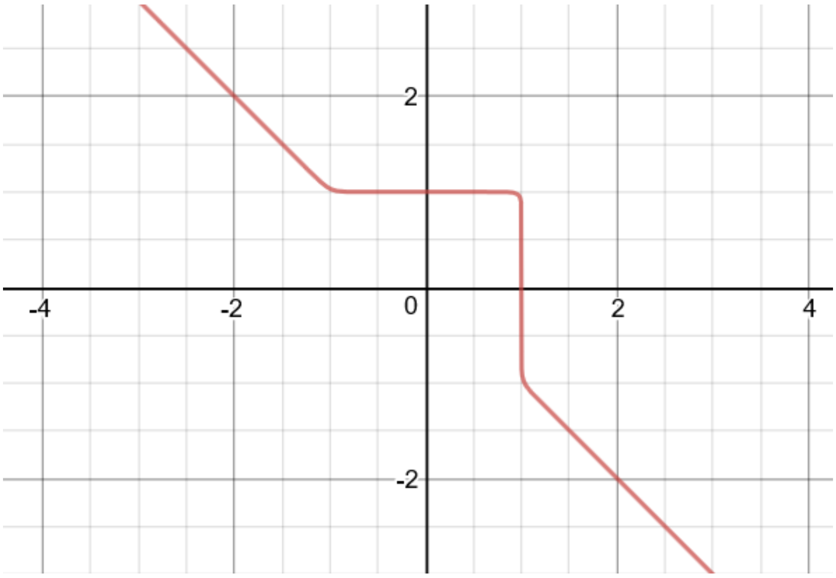
For $x < -1, x^n$ can be made as large and negative as we like (as n is odd), by making n large enough. Then $x^n - y^n = 1 \Rightarrow y \to x$ as $n \to \infty$.

Finally, $x = -1 \Rightarrow y^n = -2 \Rightarrow y \to -1$ as $n \to \infty$.



(ii) $x^n + y^n = 1$; odd n

similar reasoning can be applied



(iii) $x^n + y^n = 1$; even n

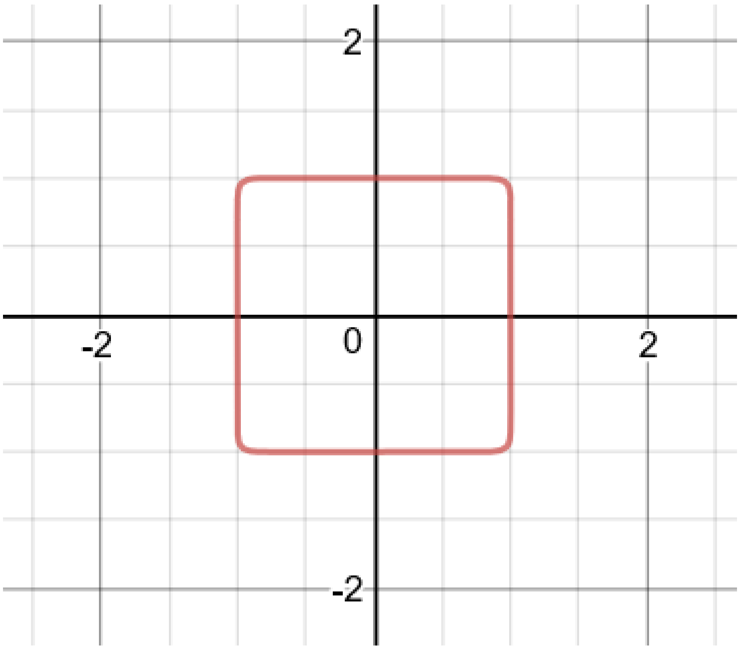
Curve has symmetry about the y-axis (replacing x with $-x$) and about the x-axis (replacing y with $-y$).

As n is even, $x^n + y^n = 1$ has no solution for $|x| > 1$ *or* $|y| > 1$.

Curve passes through $(0, 1), (0, -1), (1, 0)$ & $(-1, 0)$.

When $|x| < 1, x^n$ can be made as small as we like, by making n large enough. Then $x^n + y^n = 1 \Rightarrow y \to \pm 1$ as $n \to \infty$.

Similarly, for $|y| < 1, x \to \pm 1$ as $n \to \infty$.



[This is a straightened out version of the circle $x^2 + y^2 = 1$]

(iv) $x^n - y^n = 1$; even n

Once again, the curve has symmetry about the x and y-axes.

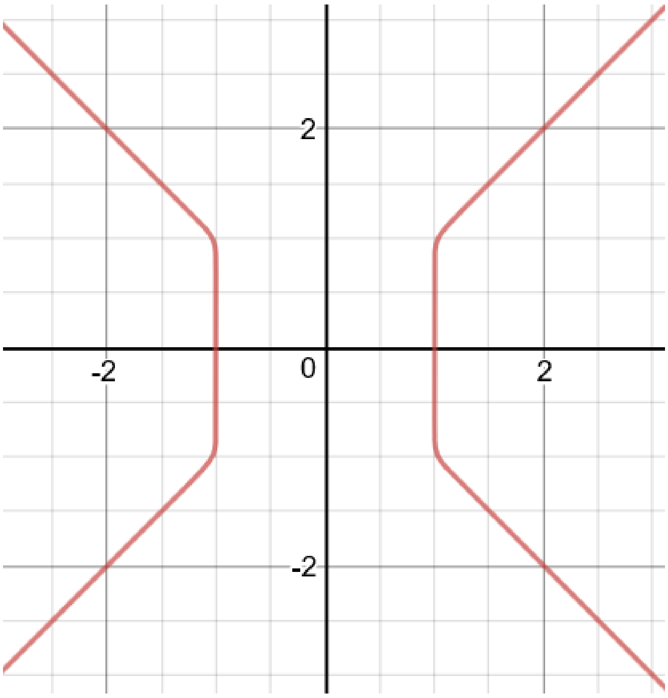
For large $|x|$ & $|y|$, $y^n = x^n - 1 \Rightarrow y \approx \pm x$; ie $y = \pm x$ are asymptotes.

Curve passes through $(1, 0)$ and $(-1, 0)$, but doesn't cross the y-axis.

When $|x| < 1, x^n$ can be made as small as we like, by making n large enough. Then $x^n - y^n = 1 \Rightarrow$ no solution.

Similarly, for $|y| < 1$, $x \to 1$ as $n \to \infty$.

For $x > 1$, x^n can be made as large as we like, by making n large enough. Then $x^n - y^n = 1 \Rightarrow y \to x$ as $n \to \infty$.



[This is a straightened out version of the rectangular hyperbola $x^2 - y^2 = 1$]