STEP Exercises - Counting (sol'ns) (11 pages; 21/9/18)

(1) 3 bananas, 4 apples and 5 oranges are to be arranged in a row. In how many ways can this be done, assuming that the bananas etc are indistinguishable?

Solution

There are 12! ways of ordering the items, where the bananas etc are distinguishable. Divide by 3! to remove the duplication, as far as the bananas are concerned, and similarly for the others, to give:

$$\frac{12!}{3!4!5!} = \frac{12(11)(10)(9)(8)(7)(6)}{6(24)} = (11)(5)(9)(8)(7) = 27720$$

(2) The following books are on a bookshelf: 4 novels, 3 history books, 2 biographies and 1 dictionary. In how many ways can they be arranged if the novels have to be together, and similarly for the history books and biographies?

Solution

[Note that we treat the novels etc as being distinguishable from each other.]

There are 4! ways of arranging the items N, H, B & D. Then to allow for the 4! ways of arranging the novels etc, we multiply by 4!3!2!, to give: $4!4!3!2! = 6912$

(3) (i) 3 different sweets are to be shared amongst 5 children. In how many ways can this be done, if no child is to receive all 3 sweets?

(ii) What is the answer if the 3 sweets are indistinguishable?

Solution

(i) Method 1

There are 5^3 ways of allocating the children to the 3 sweets. Then deduct the 5 ways in which one child has all 3 sweets, to give

$5^3 - 5 = 120$

Method 2

The permissible cases are: ABC, AAB, ABA, BAA (where AAB means that sweets 1 & 2 go to child A & sweet 3 goes to child B).

The total number of possibilities is then:

$5(4)(3) + 5(4) + 5(4) + 5(4)$ [for AAB eg, there are 5 ways of choosing A, and then 4 ways of choosing B]

$= 60 + 20 + 20 + 20 = 120$

Method 3

As Method 2, but categorise the cases as just ABC, AAB (where AAB means that 1 child receives exactly 2 sweets)

The total number of possibilities is then:

$5(4)(3) +$

5 [number of ways of choosing A]

$\times$ 4 [number of ways of choosing B]

$\times$ 3 [number of ways of placing B]

$= 60 + 60 = 120$

(ii) If the 3 sweets are indistinguishable, there are 2 permissible cases:

ABC and AAB (where AAB means that 1 child receives exactly 2 sweets)

The total number of possibilities is then:

$\binom{5}{3}$ [number of ways of choosing A,B & C]

5 [number of ways of choosing A]

$\times$ 4 [number of ways of choosing B]

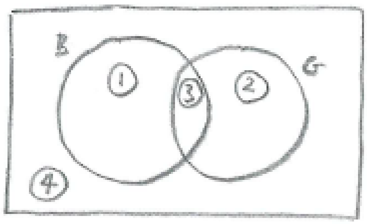
$= 10 + 20 = 30$

(4) 2 boys and 3 girls are to sit in a row. How many arrangements are there in which the 2 boys are not next to each and the 3 girls are also not next to each other?

Solution

A Venn diagram could be created to illustrate the various possibilities.

Let B denote the cases where the boys are together, and G the cases where the girls are together. There are then 4 regions in the Venn diagram, as shown below.



We are interested in region (4).

The total number of ways of arranging the 5 children is $5! = 120$. This is (1)+(2)+(3)+(4).

To find (3), where the boys are together and also the girls: Let M denote the block of boys, and F the block of girls [to avoid confusion with the B and G already used in the Venn diagram].

We can then have either MF or FM. Multiplying this figure of 2 by 2!3! in order to allow for the arrangements of the boys within M etc, gives (3)=24.

To find B = (1)+(3): consider the number of arrangements of $MF_1F_2F_3$

This gives 4! (ways of arranging these 4 items) × 2! (ways of arrangements the boys within their block); ie B=(1)+(3)=48

Similarly, $G = (2) + (3) =$ number of arrangements of $FM_1M_2 = 3!\,3! = 36$

To summarise so far,

$(1) + (2) + (3) + (4) = 120$

$(3) = 24$

$(1) + (3) = 48$

$(2) + (3) = 36$

Hence $(1) = 24, (2) = 12$ and $(4) = 120 - 24 - 12 - 24 = 60$

[We could also list the possible arrangements in the situation where the boys and girls are taken to be indistinguishable, and multiply by $2!\,3! = 12$ in each case (the number of ways of arranging the boys and girls for each of these possibilities).

Labelling each with the appropriate region from the Venn diagram:

MMFFF (3)

MFMFF (4)

MFFMF (4)

MFFFM (2)

FMMFF (1)

FMFMF (4)

FMFFM (4)

FFMMF (1)

FFMFM (4)

FFFMM (3)

This gives (1)=2, (2)=1, (3)=2, (4)=5

And multiplying each by 12 gives the figures arrived at previously.]

(5) Five poorly-behaved pupils are required to sit in the front five places in a classroom. Angus insists on sitting next to Bruce, Chantal refuses to sit next to Deborah, and Emily is happy to sit anywhere. In how many different ways can they take their seats?

A 24 B 12 C 30 D 20 E 6

Solution

Let X represent A & B.

Suppose that X is to the left of E.

Then C & D can go in the spaces shown here: _X_E_

There are 3 choices for C, and then 2 choices for D, giving 6 possibilities.

Multiply by 2, to include cases where X is to the right of E; giving 12 possibilities.

Multiply by 2 again, as X could be AB or BA, giving 24 possibilities.

Answer is A

Alternative approach

Number of ways with no constraints on C & D, where A is ahead of B (eg XABXX):

4 (ways of placing A)

$\times$ 3! (ways of placing C, D & E)

$= 24$

Including cases where B is ahead of A gives $24 \times 2 = 48$ (1)

Permutations to be excluded, with A ahead of B and C ahead of D:

ABCDE, ABECD, EABCD, CDABE, CDEAB, ECDAB

giving a total of $6 \times 2 \times 2 = 24$ to be excluded (including cases where B is ahead of A and/or D is ahead of C)   (2)

Hence, number of allowable ways $= (1) - (2) = 24$

[It is also possible to consider only situations of the form ABXXX (or BAXXX) and XABXX (or XBAXX), and multiply by 2 to cover the symmetrical situations where we start from the other end.]

(6) 6 people (labelled A-F) are to be seated round a circular table. How many seating arrangements are possible if B and E are not to sit next to each other?

Solution

First of all, note that ABCDEF and BCDEFA will (usually) be considered to be the same seating arrangement, as the table is circular.

Then, without loss of generality, we can start with B.

Method 1

Consider 3 separate cases: BXEXXX, BXXEXX & BXXXEX (these are the permissible cases). [Note that E can't be at the end, as then they would be next to B at the circular table.]

In each case, there are 4! possible arrangements.

Hence there are $3 \times 4! = 72$ possible arrangements overall.

Method 2

Starting with B, there are 5! ways of filling the remaining 5 places (including non-permissible arrangements).

From these, deduct the non-permissible arrangements, which are of the form BEXXXX or BXXXXE

There are $2 \times 4!$ of these, giving a final answer of

$5! - 2 \times 4! = 120 - 48 = 72$

(7) A 4-digit password is made up of numbers from 0 to 4, where the numbers can be repeated, but have to be ordered from largest to smallest. Show that there are 70 possible passwords.

Solution

Consider a simpler version of the problem, with just 3 numbers, each of which can be $0, 1\ or\ 2$

Then the following are possible:

$222, 221, 220, 211, 210, 200$

$111, 110, 100$

000

It may be possible to come up with a recurrence relation of some sort. Let $f(m, n)$ be the number of possibilities when we can choose between $0, 1, \dots, m$ for each digit, and there are n digits in the password.

From the above, we have $f(2,3) = f(2,2) + f(1,2) + f(0,2)$

and this can be generalised to

$f(m,n) = f(m,n-1) + f(m-1,n-1) + \cdots + f(0,n-1)$

We also note that $f(m,1) = m+1$

Applying this to the problem in question,

$f(4,4) = f(4,3) + f(3,3) + f(2,3) + f(1,3) + f(0,3)$

$= [f(4,2) + f(3,2) + \cdots + f(0,2)]$

$+[f(3,2) + f(2,2) + \cdots + f(0,2)]$

$+ \cdots + f(0,2)$

$= f(4,2) + 2f(3,2) + 3f(2,2) + \cdots + 5f(0,2)$

$= [f(4,1) + f(3,1) + \cdots + f(0,1)]$

$+2[f(3,1) + f(2,1) + \cdots + f(0,1)]$

$+ \cdots 5f(0,1)$

$= f(4,1) + (1+2)f(3,1) + (1+2+3)f(2,1)$

$+(1+2+3+4)f(1,1) + (1+2+3+4+5)f(0,1)$

$= 5 + 3(4) + 6(3) + 10(2) + 15(1)$

$= 5 + 12 + 18 + 20 + 15 = 70$

[This method can be applied to the original BMO problem, though it would become unwieldy if the length of the password exceeded 6 digits.]

Alternative (quicker) method [based on the official solutions, contained in "A Mathematical Olympiad Primer" by Geoff Smith]:

Each possibility can be represented using the following system:

3220 is represented by DXDXXDDX

and 4311 is represented by $XDXDDXXD$

For 3220, the 1st letter D means that we are dropping by 1 from the maximum of 4, and the 2nd letter X means that we have reached the value of the 1st digit; similarly the next 2 letters DX mean that we are dropping by another 1 to arrive at the 2nd digit; the 5th letter X means that we don't drop at all to arrive at the 3rd digit; the final letters DDX mean that we drop by 2 to arrive at the last digit.

For 4311, we need a D on the end to get down to 0.

In all cases there will be 4 Ds and 4 Xs, and the Ds and Xs can appear in any of the places, so that the number of possibilities is $\binom{8}{4} = 70$.

[See the official solutions for a couple of other approaches.]

(8) Given 6 pairs of twins, in how many ways can they be placed in 3 teams of 4, such that no team contains any pair of twins?

Solution

Let the 6 pairs of twins be labelled $AaBb \ldots Ff$

Define team 1 to be the team containing A.

Team 1 might, for example, be $AbCd$.

The number of ways of choosing the 3 people to go with A is

$\binom{5}{3}$ [the number of ways of choosing 3 of the 5 remaining pairs]

$\times 2^3$ [as either twin could be chosen for each pair]

For the case where team 1 is $AbCd$, define team 2 to be the team containing E. [Had E been chosen for team 1, it would have been another letter that was chosen to define team 2]

One complete selection of teams would then be:

team 1: $AbCd$

team 2: $a\ \ c\ \ \ Ef$

team 3:   B DeF

There is the following scope for choice:

$\binom{4}{2}$ ways of choosing the 2 people out of $aBcD$ to go in team 2; combined with the 2 ways of choosing either $f\ or\ F$ for team 2

Thus the overall total number of ways is:

$\binom{5}{3} \times 2^3 \times \binom{4}{2} \times 2 = 10 \times 8 \times 6 \times 2 = 960$

Note: For team 1, an alternative calculation is as follows:

The number of ways of filling the remaining 3 places in team 1 is $10 \times 8 \times 6$, if order is important; but, as order isn't important, we divide by 3!, to give 10×8, as before.