STEP - Cubic Functions (3 pages; 25/2/18)

[See Cubics - Exercises]

[based on $y = f(x) = ax^3 + bx^2 + cx + d$]

(1) Point of Inflexion

A point of inflexion occurs at the turning point of the gradient. A turning point occurs when the gradient changes sign (either from positive to negative, in the case of a maximum, or from negative to positive, in the case of a minimum). So a point of inflexion occurs when the gradient of the gradient changes sign; ie when $f''(x)$ changes sign.

$f'(x) = 3ax^2 + 2bx + c$

$f''(x) = 6ax + 2b$

$f''(x) = 0 \Rightarrow x = -\frac{b}{3a}$, and there will be a change of sign of $f''(x)$ at this point.

Thus, every cubic function possesses exactly one point of inflexion.

(2) Rotational symmetry

Without loss of generality (WLOG), as far as its shape is concerned, we can consider a cubic that has its point of inflexion at the Origin, and is therefore of the form $y = g(x) = ax^3 + cx$.

As $g(-x) = -g(x)$ [ie $g(x)$ is an odd function], there is rotational symmetry (of order 2) about the point of inflexion.

The symmetry implies that the point of inflexion is halfway between the turning points (if they exist).

[Compare with the quadratic $y = ax^2 + bx + c$, where the turning point is at $x = -\frac{b}{2a}$, which is halfway between the roots (when they exist).]

[See my solution to STEP 2, 2007, Q2 for an alternative method.]

(3) Types of Cubic

Cubics can be classified by the number of stationary points.

As $f'(x) = 3ax^2 + 2bx + c$, there will be 0, 1 or 2 stationary points, according to whether $(2b)^2 - 4(3a)c$ is negative, zero or positive; ie whether if $b^2 - 3ac$ is negative, zero or positive.

So there are 3 types:

(i) When $b^2 - 3ac < 0$

eg $y = x^3 + x$

$\frac{dy}{dx} > 0$ for all x, or $\frac{dy}{dx} < 0$ for all x (no turning points)

(ii) When $b^2 - 3ac = 0$

eg $y = x^3$

1 stationary point (at the point of inflexion); no turning points

(iii) When $b^2 - 3ac > 0$

eg $y = x^3 + x^2$

2 turning points

(4) Average of the roots

The x-coefficient of the point of inflexion can be shown to be the average of the roots of the cubic (including cases where there are complex roots).

(5) Transformations

(i) The cubic $y = x^3 + 2x^2 + x + 3$ can be sketched by translating $y = x(x^2 + 2x + 1) = x(x + 1)^2$ by $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$

(ii) The cubic $y = x^3 + 3x^2 + x + 1$ has its point of inflexion at $(-1, 2)$. In order to translate it by $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$, so that the point of inflexion is at the Origin, we write

$y + 2 = (x - 1)^3 + 3(x - 1)^2 + (x - 1) + 1$

so that $y = x^3 + x^2(-3 + 3) + x(3 - 6 + 1) - 1 + 3 - 2$

and $y = x^3 - 2x$