STEP: Algebra Methods (5 pages; 23/1/18)

including Polynomials

(1) Rearranging into the form $f(x) = 0$

(it being generally easiest to aim for a target of 0; especially where $f(x)$ is a fraction, so that only the numerator has to equal 0)

Example: Show that $\frac{sec\theta+1-tan\theta}{sec\theta+1+tan\theta} \equiv sec\theta - tan\theta$

Solution: To show that $\frac{sec\theta+1-tan\theta}{sec\theta+1+tan\theta} - (sec\theta - tan\theta) \equiv 0$:

LHS $= \frac{(sec\theta+1-tan\theta)-(sec\theta-tan\theta)(sec\theta+1+tan\theta)}{sec\theta+1+tan\theta}$

Numerator $= (sec\theta + 1 - tan\theta)$

$-(sec\theta - tan\theta)(sec\theta + tan\theta) - (sec\theta - tan\theta)$

$= (sec\theta + 1 - tan\theta) - (sec^2\theta - tan^2\theta) - (sec\theta - tan\theta)$

$= (sec\theta + 1 - tan\theta) - 1 - (sec\theta - tan\theta) = 0$

(2) Forcing into the form of the required expression

Example: Show that $\frac{sec\theta+1-tan\theta}{sec\theta+1+tan\theta} \equiv sec\theta - tan\theta$

Solution: $LHS = (sec\theta - tan\theta).\frac{(sec\theta+1-tan\theta)}{(sec\theta-tan\theta)(sec\theta+1+tan\theta)}$ (A)

Then $(sec\theta - tan\theta)(sec\theta + tan\theta) = sec^2\theta - tan^2\theta = 1$, so that

$(sec\theta - tan\theta)(sec\theta + 1 + tan\theta) = 1 + (sec\theta - tan\theta)$, and hence

$(A) = sec\theta - tan\theta$

(3) To deal with (say) 3 equations of the form

$f(x,y,z,\ldots) = 0$, $g(x,y,z,\ldots) = 0$ & $h(x,y,z,\ldots) = 0$,

where we are not interested in x, and where x can be made the subject of two of the equations (say the 1st two),

rewrite those equations as $x = A(y,z,\ldots)$ & $x = B(y,z,\ldots)$,

to obtain $A(y,z,\ldots) = B(y,z,\ldots)$ and $h(A(y,z,\ldots),y,z,\ldots) = 0$

Example (to illustrate the 1st part of the process):

Given that $\frac{bc-a}{1-c} = 7$ & $\frac{b^2c-a^2}{1-c} = 51$, show that $\frac{a+7}{b+7} = \frac{a^2+51}{b^2+51}$ (subject to any necessary conditions)

Solution

$\frac{bc-a}{1-c} = 7 \;\;\Rightarrow bc - a = 7 - 7c \Rightarrow c(b+7) = 7 + a$

$\Rightarrow c = \frac{a+7}{b+7}$ (provided $b \neq -7$) , and replacing a, b & 7 with a^2, b^2 & 51 gives $c = \frac{a^2+51}{b^2+51}$, so that $\frac{a+7}{b+7} = \frac{a^2+51}{b^2+51}$

(4) Converting from parametric to Cartesian form

(a) Make t the subject of one of the equations for x *or* y, and substitute for t in the other equation.

(b) Combine the equations for x & y in some way, so as to make t the subject (see Example (i) below)

(c) Make $f(t)$ the subject of both of the equations for x & y, and equate the two expressions, leaving a single t in the resulting equation (see Example (ii) below)

Examples

(i) $x = 2t + t^2,\ y = 2t^2 + t^3$

(ii) $x = 5t^2 - 4,\ y = 9t - t^3$

Solutions

(i) $x = 2t + t^2,\ y = 2t^2 + t^3 \Rightarrow x = t(2 + t), y = t^2(2 + t)$

So $\frac{y}{x} = t$; then $x = 2\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2$ and hence $x^3 = 2xy + y^2$

(ii) $x = 5t^2 - 4,\ y = 9t - t^3 = t(9 - t^2)$; then $t^2 = \frac{x+4}{5}$ and also $\frac{y}{t} - 9 = -t^2$; so $\frac{x+4}{5} = 9 - \frac{y}{t}$ and hence $\frac{y}{t} = 9 - \frac{x+4}{5} = \frac{45-x-4}{5} = \frac{41-x}{5}$, so that $t = \frac{5y}{41-x}$; then , substituting back into $x = 5t^2 - 4$, we have $x = 5\left(\frac{5y}{41-x}\right)^2 - 4$, and hence $(x + 4)(41 - x)^2 = 125y^2$

(5) $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

& $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

In general, for all integer $n > 1$:

$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$

[Note that, writing $f(x) = x^n - y^n, f(y) = 0$, so that $x - y$ has to be a factor.]

But $x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + \cdots - xy^{n-2} + y^{n-1})$ only for odd n. (Note the alternating signs in the 2nd bracket; consider for example $x = y = 1$.)

[Note that $x^n + y^n \geq 0$ when n is even, and $x^n + y^n = 0$ only when $x = y = 0$ (ie not for all x & y); and so there are no linear factors.]

(6) Factorisations of polynomials

(i) Factor theorem (consider factorisation of constant term)

(ii) Avoid long division (too slow)

(iii) (a) deduce one term of the divisor at a time (could set out as a table)

(b) equating coefficients

Example: Factorise $2x^3 - 33x^2 - 6x + 11$

Solution

If the factorisation is of the form $= (x + a)(2x^2 + bx + c)$,

then a must be $\pm$ a factor of 11

Applying the factor theorem this is found not to be the case.

Let $2x^3 - 33x^2 - 6x + 11 = (2x + a)(x^2 + bx + c)$,

Equating coefficients gives:

$-33 = 2b + a, -6 = 2c + ab$ & $11 = ac$

Testing the possible combinations of a & c ($\pm$ the factors of 11) shows that $a = -1, c = -11$ & $b = -16$

ie $2x^3 - 33x^2 - 6x + 11 = (2x - 1)(x^2 - 16x - 11)$

(7) $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + ac + bc)$

Other expansions such as $(a + b + c)^3 = (a^3 + b^3 + c^3) + 3(a^2b + a^2c + b^2a + b^2c + c^2a + c^2b) + 6abc$

can be found by considering symmetry and the number of combinations of each type of term.

For example, there are 3 ways of creating an a^2b term: 3[number of ways of choosing the b]$\times$ 1[number of ways of choosing two as from the remaining 2 brackets].

(8) Beware of losing a solution of an equation by dividing out a factor.

(9) Beware of spurious solutions: see STEP Problems/D/2

(10) Any rational roots of $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ are integers, if the a_i are integers. [see STEP 2011, P3, Q2 (1st part)]