

STEP 2017, P2, Q1 - Solution (X pages; 14/10/19)

(i) By Parts, $I_n = \int_0^1 x^n \arctan x \, dx$

$$= \left[\frac{x^{n+1}}{n+1} \arctan x \right]_0^1 - \int_0^1 \frac{x^{n+1}}{n+1} \left(\frac{1}{1+x^2} \right) dx \quad (\text{for } n \geq 0)$$

$$= \frac{\left(\frac{\pi}{4}\right)}{n+1} - 0 - \frac{1}{n+1} \int_0^1 \frac{x^{n+1}}{1+x^2} dx$$

so that $(n+1)I_n = \frac{\pi}{4} - \int_0^1 \frac{x^{n+1}}{1+x^2} dx$, as required

$$\text{And } I_0 = \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2} dx = \frac{\pi}{4} - \frac{1}{2} [\ln(1+x^2)]_0^1$$

$$= \frac{\pi}{4} - \frac{1}{2} \ln 2$$

$$(ii) \text{ From (i), } (n+3)I_{n+2} + (n+1)I_n = \frac{\pi}{2} - \int_0^1 \frac{x^{n+3} + x^{n+1}}{1+x^2} dx$$

$$= \frac{\pi}{2} - \int_0^1 x^{n+1} dx = \frac{\pi}{2} - \left[\frac{x^{n+2}}{n+2} \right]_0^1 = \frac{\pi}{2} - \frac{1}{n+2}$$

$$\text{So } 3I_2 + I_0 = \frac{\pi}{2} - \frac{1}{2} \text{ and } 5I_4 + 3I_2 = \frac{\pi}{2} - \frac{1}{4}$$

$$\text{Then } 5I_4 + \left(\frac{\pi}{2} - \frac{1}{2} - I_0 \right) = \frac{\pi}{2} - \frac{1}{4}$$

$$\Rightarrow 5I_4 = \frac{1}{4} + I_0 = \frac{1}{4} + \left(\frac{\pi}{4} - \frac{1}{2} \ln 2 \right), \text{ from (i)}$$

$$\Rightarrow I_4 = \frac{1}{20} (1 + \pi - 2 \ln 2)$$

$$(iii) \text{ When } n = 1, \text{ result to prove: } 5I_4 = A - \frac{1}{2} \sum_{r=1}^2 (-1)^r \cdot \frac{1}{r}$$

$$= A - \frac{1}{2} \left(-1 + \frac{1}{2} \right) = A + \frac{1}{4}; \text{ ie } I_4 = \frac{1}{20} (1 + 4A)$$

From (ii), the result is true when $\pi - 2\ln 2 = 4A$;

$$\text{ie } A = \frac{1}{4}(\pi - 2\ln 2)$$

Suppose that the result is true for $n = k$, so that

$$(4k + 1)I_{4k} = A - \frac{1}{2} \sum_{r=1}^{2k} (-1)^r \cdot \frac{1}{r} \quad (1)$$

$$\text{Target result for } n = k + 1: (4k + 5)I_{4k+4} = A - \frac{1}{2} \sum_{r=1}^{2k+2} (-1)^r \cdot \frac{1}{r}$$

$$\text{From (ii), } (4k + 5)I_{4k+4} + (4k+3) I_{4k+2} = \frac{\pi}{2} - \frac{1}{4k+4} \quad (2)$$

$$\text{and also } (4k + 3)I_{4k+2} + (4k+1) I_{4k} = \frac{\pi}{2} - \frac{1}{4k+2} \quad (3)$$

Then, (2) – (3) gives

$$(4k + 5)I_{4k+4} - (4k+1) I_{4k} = \frac{1}{4k+2} - \frac{1}{4k+4}$$

$$\text{Then, from (1), } (4k + 5)I_{4k+4} = A - \frac{1}{2} \sum_{r=1}^{2k} (-1)^r \cdot \frac{1}{r} + \frac{1}{4k+2} - \frac{1}{4k+4}$$

$$= A - \frac{1}{2} \sum_{r=1}^{2k+2} (-1)^r \cdot \frac{1}{r} + \frac{1}{2} \left\{ (-1)^{2k+1} \cdot \frac{1}{2k+1} + (-1)^{2k+2} \cdot \frac{1}{2k+2} \right\} + \frac{1}{4k+2} - \frac{1}{4k+4}$$

$$= A - \frac{1}{2} \sum_{r=1}^{2k+2} (-1)^r \cdot \frac{1}{r} + \frac{1}{2} \left\{ -\frac{1}{2k+1} + \frac{1}{2k+2} + \frac{1}{2k+1} - \frac{1}{2k+2} \right\}$$

$$= A - \frac{1}{2} \sum_{r=1}^{2k+2} (-1)^r \cdot \frac{1}{r}, \text{ as required}$$

So, if the result is true for $n = k$, then it is true for $n = k + 1$

As it is true for $n = 1$, it will therefore be true for $n = 2, 3, \dots$ and hence all integer $n \geq 1$, by the principle of induction.