STEP 2017, P1, Q12 - Solution (3 pages; 14/10/19)

(i) [Does "equal probability" mean the probabilities are the same for all the participants , or that each number has the same probability? The rest of the question implies the latter ("Instead of the numbers being equally popular", at the start of (ii)).]

[Although the organiser is picking the winning number after the participants have chosen their numbers, it is clear that it would make no difference if the winning number were chosen in advance.]

P(no one picks the winning number) $= \left(\frac{N-1}{N}\right)^N$

Expected profit $= Nc\left(\frac{N-1}{N}\right)^N + (Nc - J)(1 - \left(\frac{N-1}{N}\right)^N)$

$= NC - J(1 - \left(\frac{N-1}{N}\right)^N)$

If $2Nc = J$, Expected profit$= NC\{\left(1-\frac{1}{N}\right)^N - 1 + \left(1-\frac{1}{N}\right)^N\}$

$= NC\{2\left(1-\frac{1}{N}\right)^N - 1\}$

$\approx NC(2e^{-1} - 1) < 0$, as $e > 2$

(ii) $1 = \sum prob. = \gamma N\left(\frac{a}{N}\right) + (N - \gamma N)(\frac{b}{N})$

$\Rightarrow 1 = \gamma a + (1-\gamma)b \quad (1)$

P(no one chooses the winning number | winning number is a popular number) $= (1 - \frac{a}{N})^N$, and similarly for an unpopular number

Expected profit $= P$(winning number is a popular number)$\times$

expected profit | winning number is a popular number

$+ P$(winning number is an unpopular number)$\times$

expected profit | winning number is an unpopular number

$= \gamma\{Nc\left(1-\frac{a}{N}\right)^N + (Nc-J)(1-\left(1-\frac{a}{N}\right)^N)$

$+(1-\gamma)\{\{Nc\left(1-\frac{b}{N}\right)^N + (Nc-J)\left(1-\left(1-\frac{b}{N}\right)^N\right)\}$

$\approx \gamma(Nc-J+Je^{-a}) + (1-\gamma)(Nc-J+Je^{-b})$

$= (\gamma J)e^{-a} + \big(J(1-\gamma)\big)e^{-b} + (Nc-J)$, as required

When $\gamma = \frac{1}{8}$ & $a = 9b$, $(1) \Rightarrow 1 = \frac{a}{8} + \frac{7}{8}b \Rightarrow 8 = a + 7b$

$\Rightarrow 8 = 9b + 7b \Rightarrow b = \frac{1}{2}$ & $a = \frac{9}{2}$

If $2NC = J$, expected profit $= 2\gamma Nce^{-a} + 2Nc(1-\gamma)e^{-b} - Nc$

$= Nc(\frac{2}{8}e^{-\frac{9}{2}} + 2\left(\frac{7}{8}\right)e^{-\frac{1}{2}} - 1)$

$= \frac{Nc}{8}e^{-\frac{9}{2}}\left(2 + 14e^4 - 8e^{\frac{9}{2}}\right) = \frac{Nc}{4}e^{-\frac{9}{2}}(1 + 7e^4 - 4e^{\frac{9}{2}})$

Result to prove: $1 + 7e^4 - 4e^{\frac{9}{2}} > 0$

LHS $= 1 + e^4(7 - 4e^{\frac{1}{2}})$

Now, $7 - 4e^{\frac{1}{2}} > 0 \Leftrightarrow 7 > 4e^{\frac{1}{2}} \Leftrightarrow 49 > 16e$

As $e < 3$, $49 > 16e$, and so LHS$> 1 + 0 > 0$

Thus the expected profit is positive, as required.

Note: For the final part, the official solution includes the line

$e^{-\frac{1}{2}}(e^{-4} + 7) - 4 > \frac{7\sqrt{3}}{3} - 4$

This presumably is from

$$e^{-\frac{1}{2}}(e^{-4}+7) > 3^{-\frac{1}{2}}(3^{-4}+7) > 3^{-\frac{1}{2}}(7) = \frac{7\sqrt{3}}{3},$$

but full working would probably need to be shown in the exam.