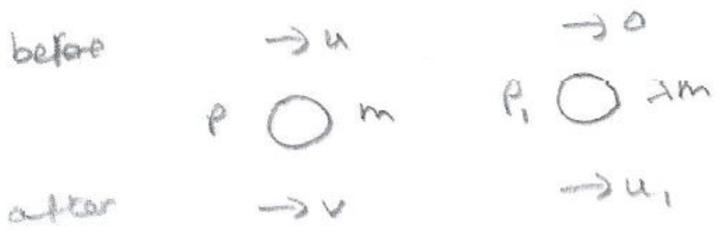
STEP 2017, P1, Q10 - Solution (4 pages; 10/10/19)

(i) The diagram shows the collision between P and P_1, where v is defined to be the velocity of P after the collision.



By conservation of momentum, $mu = mv + \lambda m u_1$,

so that $u = v + \lambda u_1$ (1)

By Newton's law of restitution (aka law of impact),

$u_1 - v = eu$ (2)

Then, adding (1) & (2), $u + eu = u_1(\lambda + 1)$,

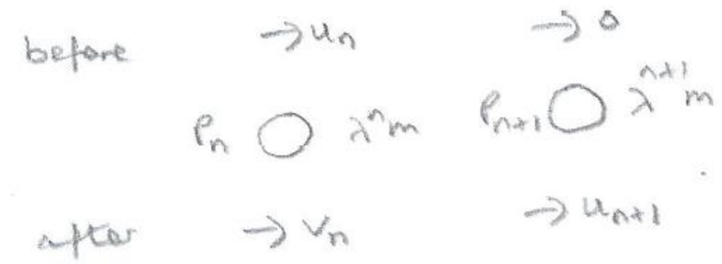
so that $u_1 = \frac{1+e}{1+\lambda}u$, as required.

As the ratio of the masses is the same for each collision,

$u_2 = \frac{1+e}{1+\lambda}u_1 = (\frac{1+e}{1+\lambda})^2 u$ etc, so that $u_n = (\frac{1+e}{1+\lambda})^n u$

Also, from (2), $v = u_1 - eu = \left(\frac{1+e}{1+\lambda} - e\right)u = \frac{1-e\lambda}{1+\lambda}u$

and, as the relation between v_n and u_n is the same as that between v and u (see diagram below),



$v_n = \frac{1-e\lambda}{1+\lambda}u_n = \frac{1-e\lambda}{1+\lambda}(\frac{1+e}{1+\lambda})^n u = \frac{(1-e\lambda)(1+e)^n}{(1+\lambda)^{n+1}}u$

(ii) Only 2 collisions $\Leftarrow v_n < v_{n+1}$

$\Leftarrow \frac{(1-e\lambda)(1+e)^n}{(1+\lambda)^{n+1}}u < \frac{(1-e\lambda)(1+e)^{n+1}}{(1+\lambda)^{n+2}}u$

$\Leftarrow 1 < \frac{1+e}{1+\lambda}$, provided that $1 - e\lambda > 0$ (noting that $\lambda > 0$ (otherwise some masses would be negative), so that $1 + \lambda > 0$, and also $1 + e > 0$)

$\Leftarrow 1 + \lambda < 1 + e \;\Leftarrow\; \lambda < e$

Then if $\lambda < e, \lambda < 1$ (as $e < 1$), so that $1 - e\lambda > 0$.

Thus $\lambda < e \Rightarrow$ only 2 collisions

[Note that we cannot write $\Leftrightarrow$ instead of $\Leftarrow$ throughout, because the condition $1 - e\lambda > 0$ depends on $\lambda < e$.]

[It would be simpler to just say that $e > \lambda \Rightarrow \frac{1+e}{1+\lambda} > 1$

$\Rightarrow v_{n+1} > v_n$, as in the official mark scheme, but the above reasoning is hopefully instructive in respect of the use of the $\Leftarrow$ symbol.]

(iii) $e = \lambda \Rightarrow u_n = (\frac{1+e}{1+\lambda})^n u = u$

and $v_n = \frac{(1-e\lambda)(1+e)^n}{(1+\lambda)^{n+1}} u = (1-e)u$

So each collision is identical, with each moving particle approaching with speed u and leaving with speed $(1-e)u$.

The original KE of P is $\frac{1}{2}mu^2$

After P_n has been struck, $P, P_1 \dots\ P_{n-1}$ are moving at speed $(1-e)u$, whilst P_n has speed u.

So the total KE at this point is:

$\frac{1}{2}m(1-e)^2u^2(1+\lambda+\lambda^2+\dots+\lambda^{n-1}) + \frac{1}{2}mu^2\lambda^n$

$= \frac{1}{2}m(1-e)^2u^2\frac{1-e^n}{1-e} + \frac{1}{2}mu^2e^n$

$= \frac{1}{2}mu^2((1-e)(1-e^n) + e^n)$

$= \frac{1}{2}mu^2(1-e+e^{n+1})$

and the fraction of the initial KE lost is

$1 - \frac{\frac{1}{2}mu^2(1-e+e^{n+1})}{\frac{1}{2}mu^2} = 1-(1-e+e^{n+1}) = e - e^{n+1}$,

which approaches e as n increases.

[The official mark scheme doesn't seem to allow for P_n having speed u.]

(iv) $\lambda e = 1 \Rightarrow u_n = (\frac{1+e}{1+\lambda})^n u = (\frac{1+e}{1+1/e})^n u = (\frac{e(1+e)}{e+1})^n u = e^n u$

and $v_n = \frac{(1-e\lambda)(1+e)^n}{(1+\lambda)^{n+1}} u = 0$ (and $v = \frac{1-e\lambda}{1+\lambda} u = 0$)

So each particle acquires a speed of $e^n u$ after its first collision, and stops after its second collision.

After P_n has been struck, it is the only particle moving, and so the fraction of the initial KE lost is

$1-\frac{\frac{1}{2}m(e^n)^2u^2}{\frac{1}{2}mu^2} = 1-e^{2n}$, which tends to 1 as $n \to \infty$

ie all KE is eventually lost.