STEP 2016, Paper 3, Q3 – Solution (3 pages; 5/6/19)

(i) $\frac{d}{dx}\left(\frac{P(x)e^x}{Q(X)}\right)=\frac{x^3-2}{(x+1)^2}e^x$

$\Rightarrow \frac{Q(x)[P'(x)e^x+P(x)e^x]-P(x)e^xQ'(x)}{[Q(x)]^2}=\frac{x^3-2}{(x+1)^2}e^x$

$\Rightarrow \frac{Q(x)[P'(x)+P(x)]-P(x)Q'(x)}{[Q(x)]^2}=\frac{x^3-2}{(x+1)^2}$ (A)

RHS is undefined when $x=-1$; hence $Q(x)$ has factor of $x-1$; otherwise $Q(-1)\neq 0$, and LHS wouldn't be undefined.

Let degrees of $P(x)$ and $Q(x)$ be p and q.

Then degree of LHS of (A) is

max$\{q+p-1, q+p, p+q-1\}-2q$

and degree of RHS is $3-2$

So $p+q-2q=1; p=q+1$, as required.

When $Q(x)=x+1$, let $P(x)=ax^2+bx+c$.

Then (A) $\Rightarrow$

$(x+1)(2ax+b+ax^2+bx+c)-(ax^2+bx+c)=x^3-2$

Equating coefficients of x^0: $b=-2$

Equating coefficients of x^1: $b+2a+c=0; 2a+c=2$

Equating coefficients of x^2: $2a+b=0; a=1; c=0$

[Equating coefficients of x^3: $a=1$]

So $P(x)=x^2-2x$

(ii) $\frac{d}{dx}\left(\frac{P(x)e^x}{Q(X)}\right) = \frac{1}{x+1}e^x$

$\Rightarrow \frac{Q(x)[P'(x)e^x+P(x)e^x]-P(x)e^xQ'(x)}{[Q(x)]^2} = \frac{1}{x+1}e^x$

$\Rightarrow \frac{Q(x)[P'(x)+P(x)]-P(x)Q'(x)}{[Q(x)]^2} = \frac{1}{x+1}$ (B)

Then $Q(x)$ has a factor of $x+1$ (in order for both sides of (B) to be undefined when $x=-1$).

Suppose that $Q(x) = (x+1)R(x)$

Then (B) $\Rightarrow$

$(x+1)R(x)[P'(x)+P(x)] - P(x)[R(x)+(x+1)R'(x)]$

$= (x+1)[R(x)]^2$

And $x=-1 \Rightarrow -P(-1)R(-1) = 0$, so that $R(x)$ has a factor of $x+1$, as $P(x)$ and $Q(x)$ have no common factors; in particular $x+1$

[So $Q(x) = (x+1)^2S(x)$, and we might be able to show that this continues indefinitely; so instead:]

Let $Q(x) = (x+1)^nT(x)$, where $T(x)$ doesn't have a factor

of $x+1$.

Then (B) $\Rightarrow$

$(x+1)^nT(x)[P'(x)+P(x)] - P(x)[n(x+1)^{n-1}T(x) +$
$(x+1)^nT'(x)]$

$= (x+1)^{2n-1}[T(x)]^2$

$$\Rightarrow$$

$\Rightarrow (x+1)[P'(x)+P(x)] - P(x)[nT(x)+(x+1)T'(x)]$

$= (x+1)^n[T(x)]^2$

And $x=-1 \Rightarrow -P(-1)nT(-1)=0$, which contradicts the assumption that $T(x)$ doesn't have a factor of $x+1$.

Thus the expression $\frac{P(x)e^x}{Q(x)}$ for the integral isn't possible.