STEP 2016, Paper 2, Q3 – Solution (2 pages; 4/6/18)

(i) [straightforward]

(ii) $a \geq 0 \Rightarrow f_n(a) \geq 1$, contradicting $f_n(a) = 0$

(iii) $f'_n(a)f'_n(b) = f_{n-1}(a)f_{n-1}(b)$, from (i)

Also $0 = f_n(a) = f_{n-1}(a) + \frac{a^n}{n!}$

and $0 = f_n(b) = f_{n-1}(b) + \frac{b^n}{n!}$

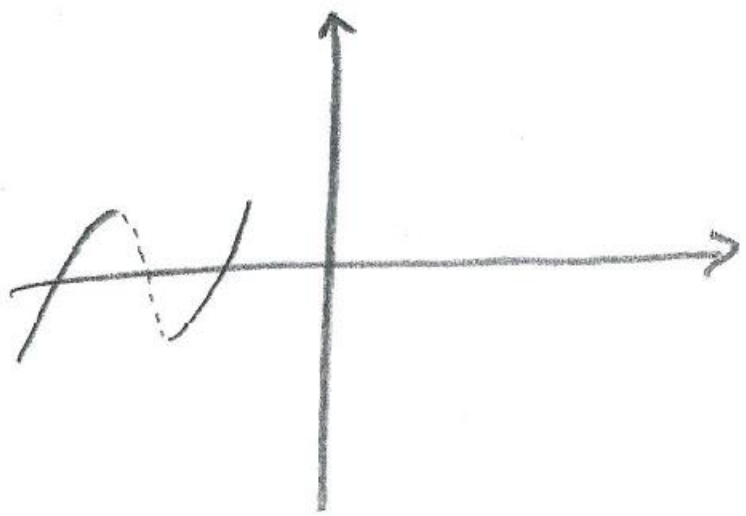
If n is even: $f_{n-1}(a) = -\frac{a^n}{n!} < 0$, and $f_{n-1}(b) < 0$ also

Then $f_{n-1}(a)f_{n-1}(b) > 0$

If n is odd: $f_{n-1}(a) = -\frac{a^n}{n!} > 0$, and $f_{n-1}(b) > 0$ also

Then $f_{n-1}(a)f_{n-1}(b) > 0$ again.

The sketch below shows the case where the gradients of $f_n(x)$ at $x = a$ and $x = b$ are both positive. As $f_n(x)$ is continuous, there must be another root c between a and b. And similarly when the gradients are both negative.



However, this implies an infinite number of roots, whereas a polynomial of order n has at most n real roots. Hence it is not possible for there to be two distinct real roots.

Also, we can show that repeated roots aren't possible:

If a root is repeated, it follows that ${f'}_n(a) = 0$.

But ${f'}_n(a) = f_{n-1}(a)$, and $0 = f_n(a) = f_{n-1}(a) + \frac{a^n}{n!}$, so that

$f_{n-1}(a) = -\frac{a^n}{n!} \neq 0$ (as $a < 0$), and ${f'}_n(a) \neq 0$

So there can be at most one real root.

If n is odd, because complex roots can only come in pairs, there must be an odd number of real roots, and hence (from the above) exactly one real root. (Alternatively, consider the number of times that the curve must cross the x-axis.)

If n is even, there must be an even number of real roots, and hence (from the above) no real roots.