STEP 2014, P1, Q2 - Solution (3 pages; 30/10/19)

(i) By Parts, $\int 1.\ln(2-x)\,dx = xln(2-x) - \int x\,.\frac{1}{2-x}\,(-1)dx$

$= xln(2-x) + \int \frac{x-2}{2-x}\,dx + \int \frac{2}{2-x}dx$

$= xln(2-x) - x - 2\ln(2-x) + c'$

$= -(2-x)\ln(2-x) + (2-x) + c,$ where $c = c' - 2,$

provided $2 - x > 0$; ie $x < 2$

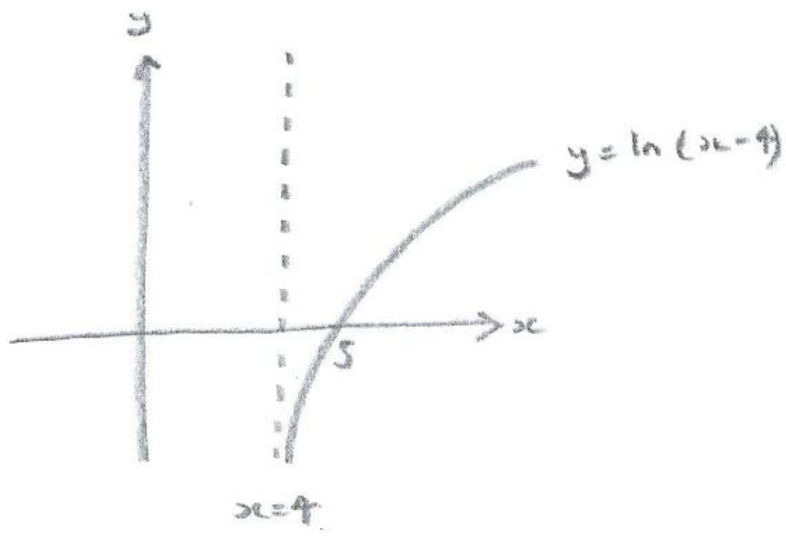
[As the official solution points out, you would be allowed to simply confirm the result by differentiation.]

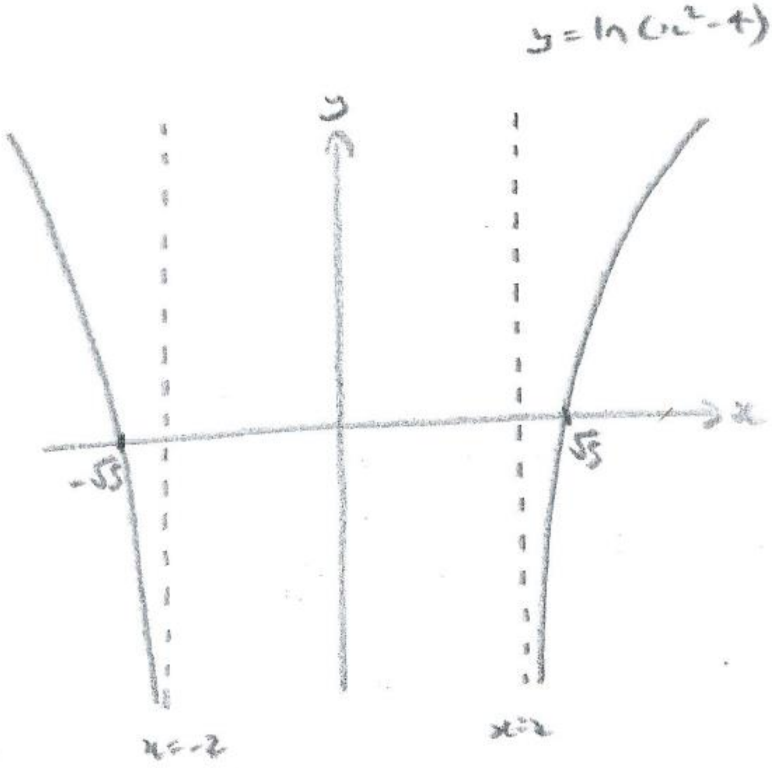
(ii) For $|x| > 2$, $y = \ln|x^2 - 4| = \ln(x^2 - 4)$ is symmetric about the y-axis. It can be based on $y = \ln(x-4)$ for $x > 4$. As $x > 1$ in this region, the effect of transforming from $y = \ln(x-4)$ to

$y = \ln(x^2 - 4)$ will be to stretch the curve in the negative

x-direction (ie so as to hug the asymptote more closely). See the diagram below. [For a given x, we are looking to the right to x^2 (which is larger than x when $x > 1$) and dragging the curve

$y = \ln(x-4)$ back to the left, to give the point $(x, \ln(x^2-4))$]



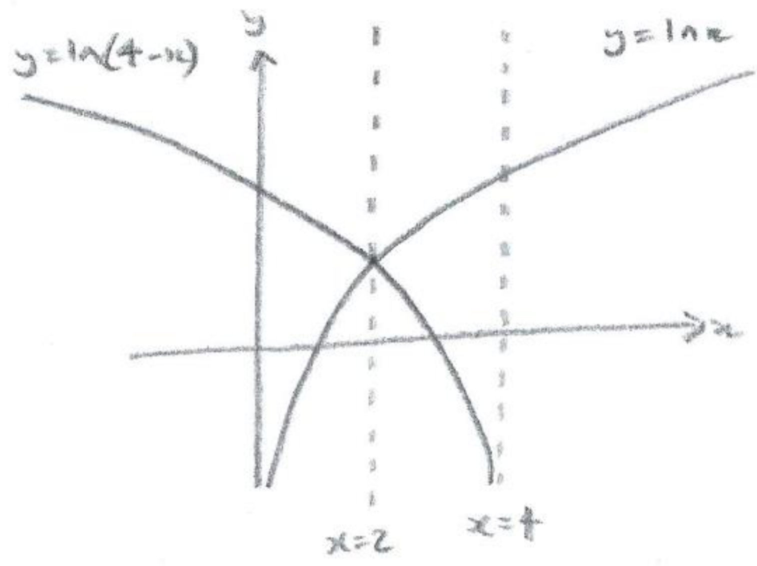


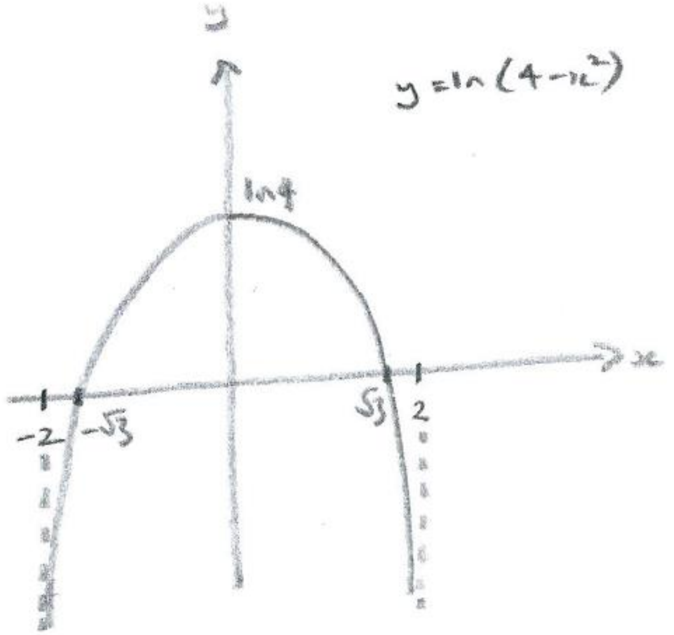
For $|x| < 2$, $y = \ln|x^2 - 4| = \ln(4 - x^2)$ will again be symmetric about the y-axis, and can be based on $y = \ln(4 - x)$ for $x < 4$, noting that the latter curve is the reflection of $y = \ln x$ in $x = 2$, as shown below.

For $0 < x < 1$, the effect of transforming from $y = \ln(4 - x)$ to

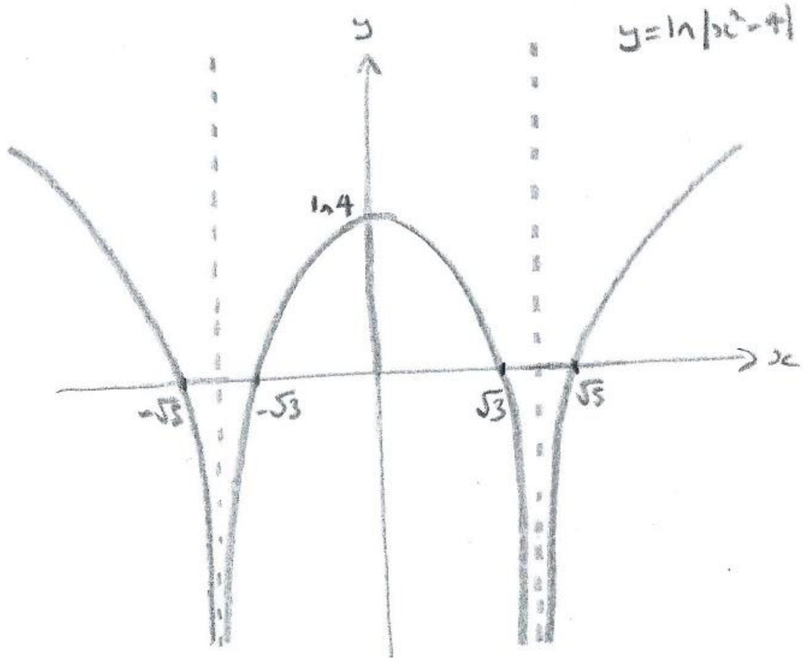
$y = \ln(4 - x^2)$ will be to stretch the curve in the positive

x-direction, whilst for $1 < x < 2$, the effect will be to stretch it in the negative x-direction.





The two curves are then combined to give $y = \ln|x^2 - 4|$:



(iii) Area $= \int_0^{\sqrt{3}} \ln(4 - x^2)\, dx = \int_0^{\sqrt{3}} \ln(2 - x) + \ln(2 + x)\, dx$

From (i), $\int \ln(2 - x)\, dx = -(2 - x)\ln(2 - x) + (2 - x)\ [+c]$

Let $u = -x$

Then $\int \ln(2+u)(-1)du = -(2+u)\ln(2+u) + (2+u)$,

and so $\int \ln(2+x)dx = (2+x)\ln(2+x) - (2+x)$

Then Area

$= [-(2-x)\ln(2-x) + (2-x) + (2+x)\ln(2+x) - (2+x)]_0^{\sqrt{3}}$

$= \{-(2-\sqrt{3})\ln(2-\sqrt{3}) - 2\sqrt{3} + (2+\sqrt{3})\ln(2+\sqrt{3})\}$

$-\{-2ln2 + 2ln2\}$

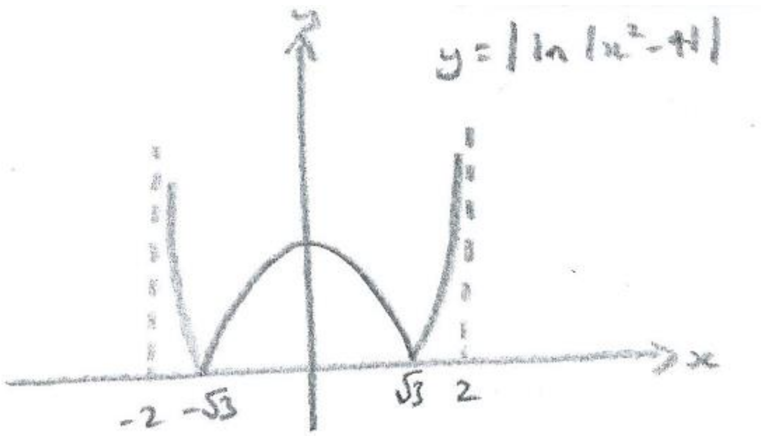
$= (2-\sqrt{3})\ln\left(\frac{1}{2-\sqrt{3}}\right) + (2+\sqrt{3})\ln(2+\sqrt{3}) - 2\sqrt{3}$

$= (2-\sqrt{3})\ln(\frac{2+\sqrt{3}}{4-3}) + (2+\sqrt{3})\ln(2+\sqrt{3}) - 2\sqrt{3}$

$= 4\ln(2+\sqrt{3}) - 2\sqrt{3}$, as required

[The official solution refers to integrating from $-\sqrt{3}$ to $\sqrt{3}$, but the question asks for the area enclosed by the positive x-axis etc.]

(iv)



From (iii), Area $= 2\{4\ln\left(2+\sqrt{3}\right) - 2\sqrt{3}\} + 2\int_{\sqrt{3}}^{2} -\ln|x^2-4|dx$

$= 8\ln\left(2+\sqrt{3}\right) - 4\sqrt{3} - 2\int_{\sqrt{3}}^{2}\ln(4-x^2)dx \quad (1)$

Now, $\int_{\sqrt{3}}^{2}\ln(4-x^2)dx = \lim\limits_{a\to 2^-}\int_{\sqrt{3}}^{a}\ln(2-x)+\ln(2+x)\,dx$

[$\int_{\sqrt{3}}^{2}\ln(4-x^2)\,dx$ is an improper integral]

$= \lim\limits_{a\to 2^-}[-(2-x)\ln(2-x) + (2-x) + (2+x)\ln(2+x)$

$-(2+x)]_{\sqrt{3}}^{a}$, from the working in (iii)

$= \{-0-4+0\}$

$-\{-\left(2-\sqrt{3}\right)\ln\left(2-\sqrt{3}\right) - 2\sqrt{3} + \left(2+\sqrt{3}\right)\ln\left(2+\sqrt{3}\right)\}$,

as $tlnt \to 0$

$= -4 - \left\{4\ln\left(2+\sqrt{3}\right) - 2\sqrt{3}\right\}$, from the working in (iii).

Then, from (1), Area

$= 8\ln\left(2+\sqrt{3}\right) - 4\sqrt{3} - 2\{-4 - \left\{4\ln\left(2+\sqrt{3}\right) - 2\sqrt{3}\right\}\}$

$= 16\ln\left(2+\sqrt{3}\right) - 8\sqrt{3} + 8$

[No answer is provided in the official solutions, so I hope this is correct!]