STEP 2013, Paper 1, Q10 – Solution (2 pages; 22/5/18)

By the work-energy principle, $\frac{1}{2}m{v_i}^2 - \frac{1}{2}m{V_i}^2 = \mu mgd$,

where m is the mass of he puck, V_i is the speed just before colliding with the wall, and μmgd is the work done against friction as the puck travels between the two barriers (since the normal reaction is mg).

[The official sol'n uses the suvat eq'n $v^2 = u^2 + 2as$, which is closely related to the above.]

And $v_{i+1} = rV_i$, so that ${v_i}^2 - \left(\frac{v_{i+1}}{r}\right)^2 = 2\mu gd$

and hence ${v_{i+1}}^2 - r^2{v_i}^2 = -2r^2\mu gd$, as required.

Then ${v_{i+2}}^2 - r^2{v_{i+1}}^2 = -2r^2\mu gd$

... ${v_{i+n}}^2 - r^2{v_{i+n-1}}^2 = -2r^2\mu gd$

Then, multiplying each row by the appropriate power of r^2, so that the intermediate terms cancel, and adding gives:

${v_{i+n}}^2 - (r^2)^{n-1}r^2{v_i}^2 = -2r^2\mu gd(1 + r^2 + \cdots + (r^2)^{n-1})$ (A)

With $v_i = v$ & $v_{i+n} = 0$ (v_{i+n} is the speed just **after** the nth collision, but as it equals 0 it will also be the speed just before the nth collision),

$(r^2)^n v^2 = 2r^2\mu gd\frac{(1-(r^2)^n)}{1-r^2}$

$\Rightarrow (r^2)^{n-1}\frac{v^2}{2\mu gd} = \frac{(1-r^{2n})}{1-r^2}$

Writing $\lambda = r^2$ for the moment, $\lambda^{n-1}k(1-\lambda) = 1 - \lambda^n$,

so that $\lambda^{n-1}\{k(1-\lambda) + \lambda\} = 1$

$\Rightarrow \lambda^{n-1} = (k(1-\lambda) + \lambda)^{-1}$

$\Rightarrow (n-1)ln\lambda = -ln(k(1-\lambda)+\lambda)$

$\Rightarrow n = 1 - \frac{ln(k(1-\lambda)+\lambda)}{ln\lambda}$

$\Rightarrow n = \frac{ln\lambda - ln(k(1-\lambda)+\lambda)}{ln\lambda}$

[since the question gives n when $r = e^{-1}$ as a single term]

$\Rightarrow n = \frac{ln(\frac{\lambda}{k(1-\lambda)+\lambda})}{ln\lambda} = \frac{ln(\frac{r^2}{k(1-r^2)+r^2})}{2lnr}$

When $r = e^{-1}$, $n = \frac{ln\left(\frac{e^{-2}}{k(1-e^{-2})+e^{-2}}\right)}{-2}$

$= \frac{1}{2} ln\left(\frac{k(1-e^{-2})+e^{-2}}{e^{-2}}\right) = \frac{1}{2} ln(k(e^2-1)+1)$, giving the required answer.

When $r = 1$, (A) becomes

${v_{i+n}}^2 - {v_i}^2 = -2\mu gdn$ (A)

so that $0 - v^2 = -2\mu gdn$

and hence $n = \frac{v^2}{2\mu gd} = k$