STEP 2012, Paper 3, Q6 – Solution (3 pages; 16/7/18)

The 1st result follows from substituting $z = x + iy$ and equating real and imaginary parts.

$(2x + p)y = 0 \Rightarrow$ either $p = -2x$ or $y = 0$

From $x^2 - y^2 + px + 1 = 0$, $y = 0 \Rightarrow p = -\frac{x^2+1}{x}$ and $x \neq 0$, (as $x = 0, y = 0$ doesn't satisfy $x^2 - y^2 + px + 1 = 0$).

$x^2 - y^2 + px + 1 = 0$, $p = -2x \Rightarrow x^2 + y^2 = 1$; ie the circle of radius 1, with centre the Origin

$y = 0$ with $x \neq 0$ is the real axis with the origin missing

For $pz^2 + z + 1 = 0$, let $z = x + iy$, so that

$p(x + iy)^2 + (x + iy) + 1 = 0$

Equating real and imaginary parts then gives

$p(x^2 - y^2) + x + 1 = 0$ (1)

and $2pxy + y = 0 \Rightarrow y(2px + 1) = 0$ (2)

Then (2)$\Rightarrow$ either (a) $y = 0$ or (b) $p = -\frac{1}{2x}$ $(x \neq 0)$

(b) & (1) $\Rightarrow x^2 - y^2 - 2x^2 - 2x = 0$

$\Rightarrow x^2 + y^2 + 2x = 0 \Rightarrow (x + 1)^2 + y^2 = 1$

ie a circle, centre $(-1,0)$ and radius 1

(a) & (1) $\Rightarrow px^2 + x + 1 = 0 \Rightarrow p = -\frac{(x+1)}{x^2}$

Thus x can take any value except 0, with $y = 0$;

ie the real axis excluding the Origin

For $pz^2 + p^2z + 2 = 0$, let $z = x + iy$, so that

$p(x + iy)^2 + p^2(x + iy) + 2 = 0$

Equating real and imaginary parts then gives

$px^2 - py^2 + p^2x + 2 = 0$ (1)

and $2pxy + p^2y = 0 \Rightarrow yp(2x + p) = 0$ (2)

From (2), either $(a)\ y = 0$ or (b) $p = -2x$

($p = 0$ doesn't satisfy (1))

(b) & (1) $\Rightarrow -2x^3 + 2xy^2 + 4x^3 + 2 = 0$

$\Rightarrow x^3 + xy^2 + 1 = 0 \Rightarrow y^2 = -\frac{(x^3+1)}{x}$ (3)

(a) & (1) $\Rightarrow xp^2 + x^2p + 2 = 0$ (4)

real $p \Rightarrow x^4 - 8x \geq 0$

either $x = 0$: this doesn't satisfy (4)

or $x > 0 \Rightarrow x^3 - 8 \geq 0 \Rightarrow x \geq 2$

or $x < 0 \Rightarrow x^3 - 8 \leq 0 \Rightarrow x < 0$

Thus the locus is given by $y^2 = -\frac{(x^3+1)}{x}$, as well as the real axis excluding $0 \leq x < 2$

To sketch $y^2 = -\frac{(x^3+1)}{x}$:

(i) symmetry about the x axis

(ii) asymptote of $x = 0$

(iii) $x = 0 + \delta \Rightarrow y^2 < 0$; ie not part of domain

(iv) $y^2 \geq 0 \Rightarrow \frac{(x^3+1)}{x} \leq 0 \Rightarrow x < 0$ & hence $x^3 + 1 \geq 0$,

so that $-1 \leq x < 0$

(v) $2y\frac{dy}{dx} = -\frac{x(3x^2)-(x^3+1)}{x^2}$

$x = -1 \Rightarrow RHS = 3$, so that $\frac{dy}{dx} = \infty$, as $y = 0$