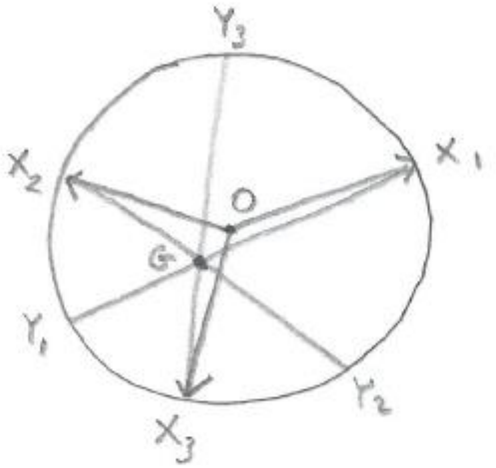
STEP, 2012, Paper 2, Q7 - Solutions (15/7/18; 2 pages)



[in fact, the diagram doesn't help much with the solution]

$\frac{|GY_1|}{|GX_1|} = \lambda$

$\overrightarrow{OY_1} = \overrightarrow{OG} + \overrightarrow{GY_1} = \overrightarrow{OG} - \lambda_1\overrightarrow{GX_1} = \overrightarrow{OG} - \lambda_1(\overrightarrow{GO} + \overrightarrow{OX_1})$

$= (1 + \lambda_1)\overrightarrow{OG} - \lambda_1\underline{x}_1 = \frac{1}{3}(1 + \lambda_1)(\underline{x}_1 + \underline{x}_2 + \underline{x}_3) - \lambda_1\underline{x}_1$

$= \frac{1}{3}\{(1 - 2\lambda_1)\underline{x}_1 + (1 + \lambda_1)(\underline{x}_2 + \underline{x}_3)\}$, as required

[We haven't yet used the fact that the radius of the circle is 1:]

As $|\overrightarrow{OY_1}| = 1$ and $|\overrightarrow{OY_1}|^2 = \overrightarrow{OY_1}\,.\overrightarrow{OY_1}$:

$\overrightarrow{OY_1}\,.\overrightarrow{OY_1} = \frac{1}{9}\{(1 - 2\lambda_1)\underline{x}_1 + (1 + \lambda_1)(\underline{x}_2 + \underline{x}_3)\}.$

$\{(1 - 2\lambda_1)\underline{x}_1 + (1 + \lambda_1)(\underline{x}_2 + \underline{x}_3)\}$

so that $9 = (1 - 2\lambda_1)^2\underline{x}_1.\underline{x}_1 + (1 + \lambda_1)^2(\underline{x}_2 + \underline{x}_3).(\underline{x}_2 + \underline{x}_3)$

$+2(1 - 2\lambda_1)(1 + \lambda_1)\underline{x}_1.(\underline{x}_2 + \underline{x}_3)$

$\Rightarrow 9 = (1 - 2\lambda_1)^2 + (1 + \lambda_1)^2(2 + 2\alpha)$

$+2(1 - 2\lambda_1)(1 + \lambda_1)(\gamma + \beta)$

(since $\underline{x}_1.\underline{x}_1 = 1$)

[At this point the H&A conveniently spots that a factor of $(1 + \lambda_1)$ can be obtained from $(1 - 2\lambda_1)^2 - 9$. With hindsight, it is possibly worth looking for something like this.]

$\Rightarrow {\lambda_1}^2(4 + 2 + 2\alpha - 4\gamma - 4\beta) + \lambda_1(-4 + 4 + 4\alpha - 2\gamma - 2\beta)$

$+1 + 2 + 2\alpha + 2\gamma + 2\beta - 9 = 0$

$\Rightarrow {\lambda_1}^2(6 + 2\alpha - 4\gamma - 4\beta) + \lambda_1(4\alpha - 2\gamma - 2\beta)$

$+2\alpha + 2\gamma + 2\beta - 6 = 0$

$\Rightarrow {\lambda_1}^2(3 + \alpha - 2\gamma - 2\beta) + \lambda_1(2\alpha - \gamma - \beta) + \alpha + \gamma + \beta - 3 = 0$

At this point it is natural to wonder if we have made a small mistake somewhere (a couple of the coefficients look encouraging though).

Using the given value of λ_1 though, we can however attempt a factorisation as follows:

$(\lambda_1 + k)([3 + \alpha - 2\beta - 2\gamma]\lambda_1 - [3 - \alpha - \beta - \gamma]) = 0$

and fortunately this works, with $k = 1$.

Thus $\lambda_1 = \frac{3-\alpha-\beta-\gamma}{3+\alpha-2\beta-2\gamma}$ (since we are told that $\lambda_1 > 0$).

By symmetry, similar expressions exist for λ_2 & λ_3.

Then $\frac{GX_1}{GY_1} + \frac{GX_2}{GY_2} + \frac{GX_3}{GY_3} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}$

$= \frac{3+\alpha-2\beta-2\gamma}{3-\alpha-\beta-\gamma} + \frac{3+\beta-2\alpha-2\gamma}{3-\alpha-\beta-\gamma} + \frac{3+\gamma-2\alpha-2\beta}{3-\alpha-\beta-\gamma}$

$= \frac{9-3\alpha-3\beta-3\gamma}{3-\alpha-\beta-\gamma} = 3$