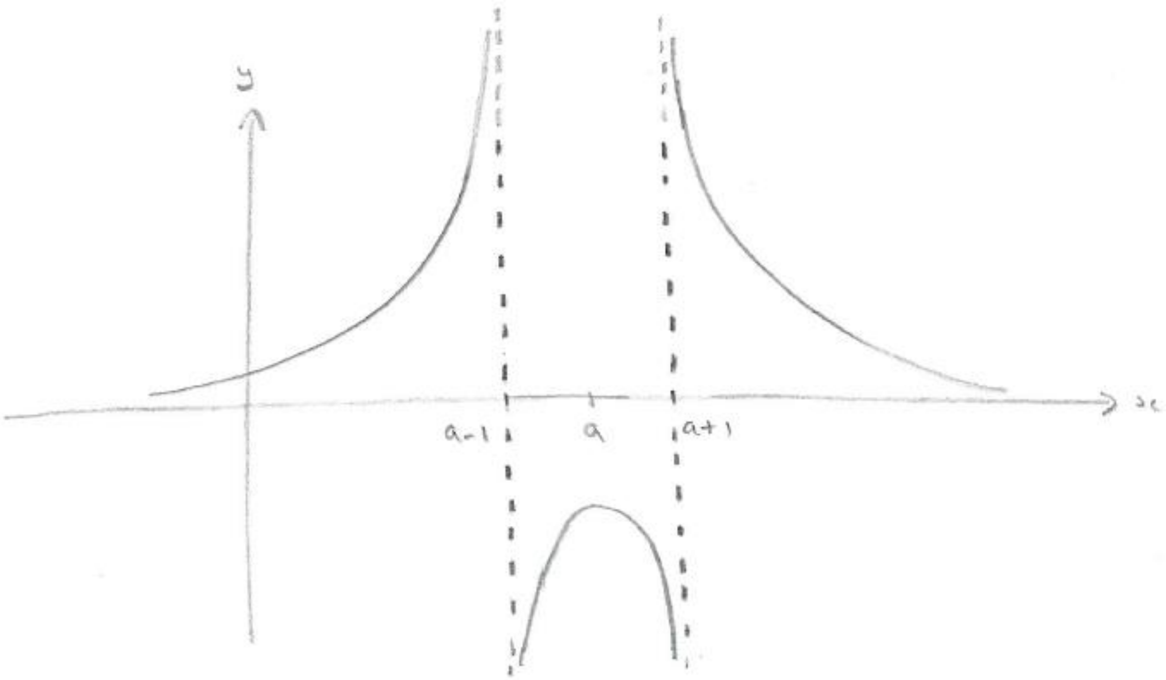
STEP, 2012, Paper 2, Q5 - Solutions (15/7/18; 4 pages)

(i) $f(x) = \frac{1}{(x-a)^2-1} = \frac{1}{(x-a-1)(x-a+1)}$

Vertical asymptotes: $x = a - 1$ & $x = a + 1$

[symmetry about $x = a$]

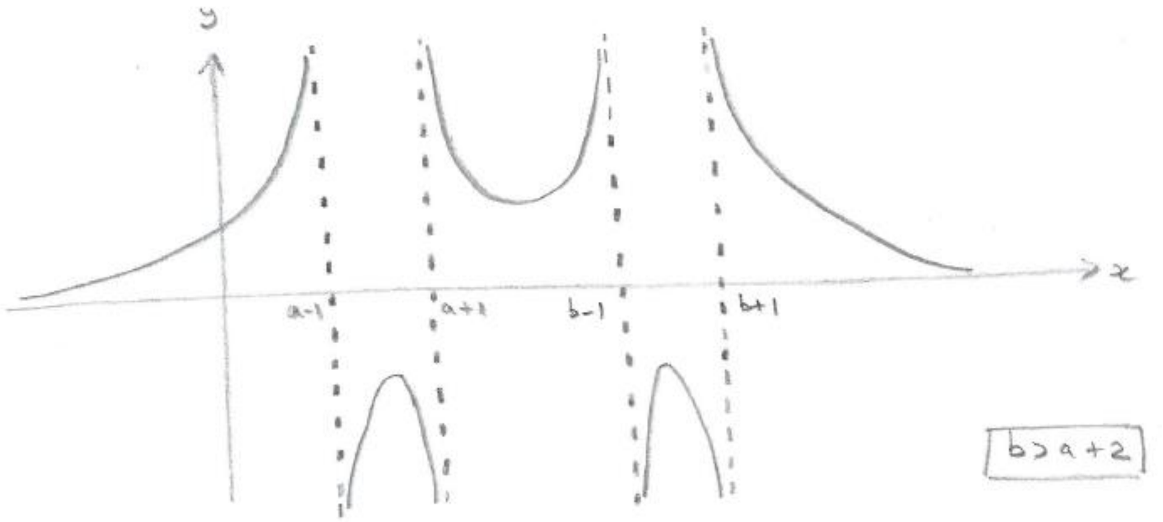
Sketch for $a > 1$ (for other values the graph is just shifted to the left):



(ii) There are now asymptotes at:

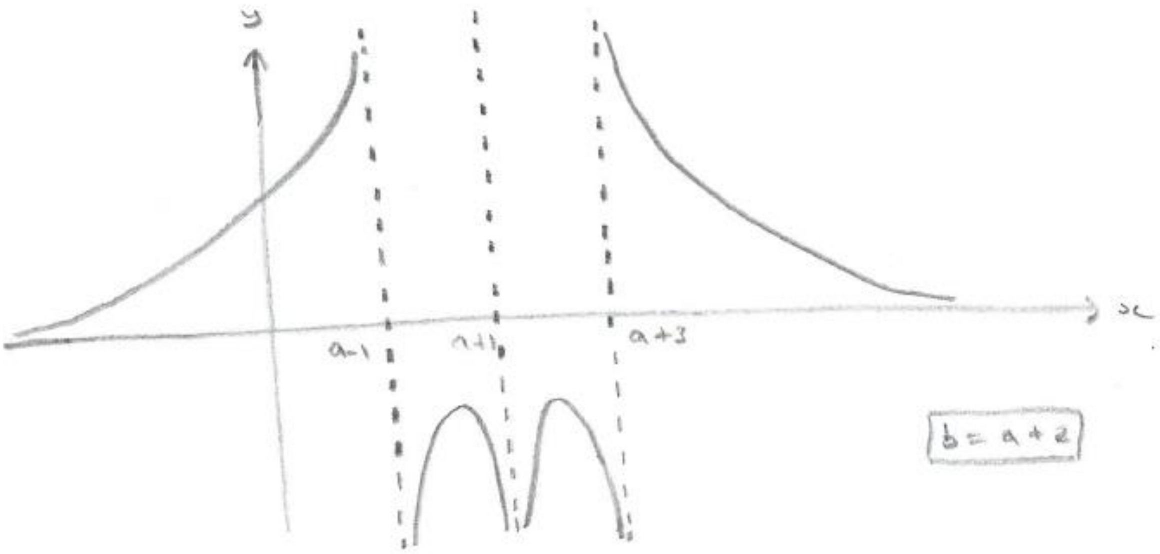
$x = a - 1$, $x = a + 1$, $x = b - 1$ & $x = b + 1$ when $b > a + 2$

and $x = a - 1$, $x = a + 1$ & $x = a + 3$ when $b = a + 2$



[Note that there is symmetry about the midpoint of $a + 1$ &

$b - 1$; ie at $\frac{1}{2}(a + b)$; though there is no indication of the position of the two maxima.]



[The repeated factor of $(x - a - 1)$ means that the asymptote is approached at the negative end for values on both sides of the asymptote. Again, there is symmetry about $x = a + 1$.]

To find the stationary points:

$$g'(x) = -[(x - a)^2 - 1]^{-2}(2)(x - a)[(x - b)^2 - 1]^{-1}$$

$$-[(x-b)^2-1]^{-2}(2)(x-b)[(x-a)^2-1]^{-1}$$

$$= -2\,\frac{(x-a)[(x-b)^2-1]+(x-b)[(x-a)^2-1]}{[(x-a)^2-1]^2[(x-b)^2-1]^2}$$

$g'(x) = 0$

$\Rightarrow (x-a)[(x-b)^2-1]+(x-b)[(x-a)^2-1] = 0$ (A)

[We thus have to solve a cubic. It is a fair bet that it will factorise conveniently, but as an alternative to hoping that an obvious factor will emerge, we can take advantage of symmetry:]

In the more general case when $b > a+2$, the symmetry about $x = \frac{1}{2}(a+b)$ means that this is one of the roots.

This implies a factor of $2x-a-b$.

From (A),

$(x-a)(x-b)\{x-b+x-a\}-(x-a)-(x-b) = 0$

$\Rightarrow (2x-a-b)\{(x-a)(x-b)-1\} = 0$

[H&A contains a typo in the line "and setting the numerator =0 …",

"… $+[x-a+x-b] = 0$"

should read "… $-[x-a+x-b] = 0$"]

$\Rightarrow (2x-a-b)\{x^2-(a+b)x+ab-1\} = 0$

$\Rightarrow x = \frac{1}{2}(a+b)$ or $\frac{(a+b)\pm\sqrt{(a+b)^2-4(ab-1)}}{2} = \frac{a+b\pm\sqrt{(a-b)^2+4}}{2}$

[Alternatively, had the cubic been harder to factorise, it would be possible to make the substitution $u = x-\frac{1}{2}(a+b)$, which would then lead to a factor of u (corresponding to a root of $u=0$).]

When $b = a + 2$, $\frac{1}{2}(a + b) = a + 1$, but $g(x)$ is not defined for $x = a + 1$, so this stationary point can be excluded.