STEP, 2012, Paper 2, Q3 - Solutions (15/7/18; 3 pages)

[The following approach is given in case you "failed to notice the obvious result that $\frac{1}{t} = \sqrt{x^2+1} - x$" (or why it might be needed) (Examiners' Report). See alternative method later on.]

Let $t = x + \sqrt{1+x^2}$

so that $dt = [1 + \frac{1}{2}(1+x^2)^{-\frac{1}{2}}(2x)]dx$

$x = 0 \Rightarrow t = 1; x = \infty \Rightarrow t = \infty$

So $\int_0^\infty f\left(x+\sqrt{1+x^2}\right)dx = \int_1^\infty \frac{f(t)}{1+x(1+x^2)^{-\frac{1}{2}}}\,dt$

$\frac{1}{1+x(1+x^2)^{-\frac{1}{2}}} = \frac{\sqrt{1+x^2}}{\sqrt{1+x^2}+x}$

[the idea now is to 'force' this expression into the form $\frac{1}{2}(1+\frac{1}{t^2})$]

$= (\frac{1}{2})\frac{2\sqrt{1+x^2}}{\sqrt{1+x^2}+x} = (\frac{1}{2})\frac{(\sqrt{1+x^2}+x+\sqrt{1+x^2}-x)}{\sqrt{1+x^2}+x}$

$= \left(\frac{1}{2}\right)\left(1+\frac{\sqrt{1+x^2}-x}{\sqrt{1+x^2}+x}\right) = \left(\frac{1}{2}\right)(1+\frac{(\sqrt{1+x^2}-x)(\sqrt{1+x^2}+x)}{t^2})$

$= \left(\frac{1}{2}\right)(1+\frac{(1+x^2)-x^2}{t^2})$

$= \left(\frac{1}{2}\right)(1+\frac{1}{t^2})$

and hence $\int_1^\infty \frac{f(t)}{1+x(1+x^2)^{-\frac{1}{2}}}\,dt = \frac{1}{2}\int_1^\infty \left(1+\frac{1}{t^2}\right)f(t)dt$, as required

Alternative method

The 'obvious' result mentioned in the Hints & Answers and Examiners' Report is used to make x the subject of the relation

$t = x + \sqrt{1+x^2}$. This can also be done as follows:

$t - x = \sqrt{1+x^2} \Rightarrow (t-x)^2 = 1 + x^2$

$\Rightarrow t^2 - 2tx = 1$

$\Rightarrow x = \frac{t^2-1}{2t} = \frac{t}{2} - \frac{1}{2t}$

This gives $dx = \left(\frac{1}{2} + \frac{1}{2t^2}\right) dt = \left(\frac{1}{2}\right)(1 + \frac{1}{t^2})dt$, from which the required result follows.

Noting that $(x + \sqrt{1+x^2})^2 = 2x^2 + 1 + 2x\sqrt{x^2+1}$,

We have $f(t) = \frac{1}{t^2}$ and so

$\int_0^\infty \frac{1}{2x^2+1+2x\sqrt{x^2+1}} dx = \frac{1}{2}\int_1^\infty (1 + \frac{1}{t^2}) \frac{1}{t^2} dt$

$= \frac{1}{2}\int_1^\infty \frac{1}{t^2} + \frac{1}{t^4} dt = \frac{1}{2}\left[-\frac{1}{t} - \frac{1}{3t^3}\right]_1^\infty$

$= \frac{1}{2}\left(0 - \left[-1 - \frac{1}{3}\right]\right) = \frac{1}{2} . \frac{4}{3} = \frac{2}{3}$

Let $J = \int_0^{\frac{\pi}{2}} \frac{1}{(1+sin\theta)^3} d\theta$ and $x = tan\theta$

Then $dx = sec^2\theta \, d\theta$, so that $d\theta = \frac{1}{1+x^2} dx$

and $sin\theta = \sqrt{1 - \frac{1}{sec^2\theta}} = \sqrt{1 - \frac{1}{1+x^2}} = \sqrt{\frac{x^2}{1+x^2}} = \frac{x}{\sqrt{1+x^2}}$

Also, when $\theta = 0, x = 0$ and when $\theta = \frac{\pi}{2}, x = \infty$

Then $J = \int_0^\infty \frac{(1+x^2)^{\frac{3}{2}}}{(1+x^2)\left(\sqrt{1+x^2}+x\right)^3}\,dx = \int_0^\infty \frac{\sqrt{1+x^2}}{\left(\sqrt{1+x^2}+x\right)^3}\,dx$

[The integrand now needs to be expressed as a function of t]

From the alternative method above, $x = \frac{t^2-1}{2t}$,

so that $\sqrt{1+x^2} = t - x = t - \frac{t^2-1}{2t} = \frac{2t^2-t^2+1}{2t} = \frac{t^2+1}{2t}$

Then $J = \int_0^\infty \frac{t^2+1}{2t^4}\,dx = \frac{1}{2}\int_1^\infty \left(1+\frac{1}{t^2}\right)\left(\frac{t^2+1}{2t^4}\right)dt$

$= \frac{1}{4}\int_1^\infty t^{-2} + t^{-4} + t^{-4} + t^{-6}\,dt$

$= \frac{1}{4}\left[\frac{t^{-1}}{-1} + \frac{2t^{-3}}{-3} + \frac{t^{-5}}{-5}\right]_1^\infty$

$= \frac{1}{4}\left(0 - \left[-1 - \frac{2}{3} - \frac{1}{5}\right]\right)$

$= \frac{1}{4}\left(1 + \frac{2}{3} + \frac{1}{5}\right) = \frac{1}{60}(15 + 10 + 3) = \frac{28}{60} = \frac{7}{15}$