

STEP 2012, Paper 1, Q8 – Solution (2 pages; 15/6/18)

Please note: The official 2012 "Hints and Answers" are mainly hints rather than answers, so the following sol'n hasn't been checked against anything.

(i) Substituting $y = xv$ into $xy \frac{dy}{dx} + y^2 - 2x^2 = 0$ gives

$$x(xv) \left(v + x \frac{dv}{dx} \right) + (xv)^2 - 2x^2 = 0$$

$$\Rightarrow xv \frac{dv}{dx} + 2v^2 - 2 = 0 \text{ (as } x \neq 0), \text{ as required}$$

Separating variables,

$$\int \frac{v}{2(1-v^2)} dv = \int \frac{1}{x} dx$$

$$\Rightarrow -\frac{1}{4} \ln(1-v^2) = \ln x + \ln A$$

$$\Rightarrow \ln(1-v^2) = -4 \ln(Ax) = \ln \left(\frac{B}{x^4} \right)$$

$$\Rightarrow 1-v^2 = \frac{B}{x^4}$$

$$\Rightarrow B = x^4 \left(1 - \frac{y^2}{x^2} \right) = x^2(x^2 - y^2)$$

$$\Rightarrow x^2(y^2 - x^2) = C$$

(ii) [The only clue here is that $x \neq 0$ again. Although it may seem too obvious, the substitution $y = xv$ is in the spirit of the STEP questions: the examiners don't usually set traps. The first step in a solution is often straightforward; complications may arise later.]

Substituting $y = xv$ into $y \frac{dy}{dx} + 6x + 5y = 0$ gives

$$xv \left(v + x \frac{dv}{dx} \right) + 6x + 5xv = 0$$

$$\Rightarrow v^2 + vx \frac{dv}{dx} + 6 + 5v = 0$$

$$\Rightarrow -vx \frac{dv}{dx} = 6 + 5v + v^2$$

$$\Rightarrow -\int \frac{v}{6+5v+v^2} dv = \int \frac{1}{x} dx$$

$$\Rightarrow -\int \frac{v+\frac{5}{2}}{6+5v+v^2} dv + \frac{5}{2} \int \frac{1}{\left(v+\frac{5}{2}\right)^2 - \frac{25}{4} + 6} dv = \ln x + \ln A$$

$$\Rightarrow -\frac{1}{2} \ln(6 + 5v + v^2) + \frac{5}{2} \int \frac{1}{\left(v+\frac{5}{2}\right)^2 - \frac{1}{4}} dv = \ln(Ax)$$

$$\Rightarrow -\ln\{(v+2)(v+3)\} + 5 \int \frac{1}{v+\frac{5}{2}-\frac{1}{2}} - \frac{1}{v+\frac{5}{2}+\frac{1}{2}} dv = 2\ln(Ax)$$

$$\Rightarrow \ln(Bx^2) + \ln\{(v+2)(v+3)\} - 5\ln(v+2) + 5\ln(v+3) = 0$$

$$\Rightarrow \ln \left\{ \frac{Bx^2(v+2)(v+3)(v+3)^5}{(v+2)^5} \right\} = 0$$

$$\Rightarrow \frac{Bx^2(v+2)(v+3)(v+3)^5}{(v+2)^5} = 1$$

$$\Rightarrow Bx^2(v+3)^6 = (v+2)^4$$

$$\Rightarrow Cx(v+3)^3 = (v+2)^2$$

$$\Rightarrow Cx\left(\frac{y}{x} + 3\right)^3 = \left(\frac{y}{x} + 2\right)^2$$

$$\Rightarrow C(y+3x)^3 = (y+2x)^2$$