STEP 2012, Paper 1, Q13 – Solution (3 pages; 15/6/18)

Please note: The official 2012 "Hints and Answers" are mainly hints rather than answers, so the following sol'n hasn't been checked against anything.

Expected number of different digits

$= \sum_{r=1}^{5} rP(r\ different\ digits|only\ digits\ 1\ to\ 5\ chosen)$

where $P(r\ different\ digits|only\ digits\ 1\ to\ 5\ chosen)$

$= \frac{P(only\ digits\ 1\ to\ 5\ chosen\ \&\ r\ different\ digits)}{P(only\ digits\ 1\ to\ 5\ chosen)}$

$P(only\ digits\ 1\ to\ 5\ chosen)$

$= P(1st\ digit\ is\ 1\ to\ 5) \times P(2nd\ digit\ is\ 1\ to\ 5) \times \ldots$

$= \frac{5}{9} \times \left(\frac{5}{10}\right)^4 = \frac{5}{9(16)}$

$P(only\ digits\ 1\ to\ 5\ chosen\ \&\ the\ same\ digits) = \frac{5}{90000}$

To find the number of ways of choosing numbers, using 2 digits from 1 to 5:

eg (a) 23233 or (b) 23333

$For\ (a)$: 5_{C_2} (number of ways of selecting pairs such as 2 & 3)

$\times\ 5_{C_2}$ $(number\ of\ ways\ of\ placing\ the\ 2s(eg))$

$\times$ 2 (as either 2 or 3 (eg) could be in the minority)

$= 10 \times 10 \times 2 = 200$

$For\ (b)$: 5_{C_2} (number of ways of selecting pairs such as 2 & 3)

$\times\ 5$ $(number\ of\ ways\ of\ placing\ the\ 2(eg))$

$\times$ 2 (as either 2 or 3 (eg) could be in the minority)

$= 10 \times 5 \times 2 = 100$

So $P(only\ digits\ 1\ to\ 5\ chosen\ \&\ 2\ different\ digits)$

$= \frac{200+100}{90000} = \frac{300}{90000}$

To find the number of ways of choosing numbers, using 3 digits from 1 to 5:

eg (a) 23444 or (b) 23344

$For\ (a)$: 5_{C_3} (no. of ways of selecting 3 numbers such as 2,3 & 4)

$\times 5 \times 4$ ($number\ of\ ways\ of\ placing\ the\ 2\ \&\ 3(eg)$)

$\times 3$ (as 2, 3 or 4 could be in the majority)

$= 10 \times 20 \times 3 = 600$

$For\ (b)$: 5_{C_3} (no. of ways of selecting 3 numbers such as 2,3 & 4)

$\times 5$ ($number\ of\ ways\ of\ placing\ the\ 2\ (eg)$)

$\times 4_{C_2}$($number\ of\ ways\ of\ placing\ the\ 3s\ \&4s(eg)$)

$\times 3$ (as 2, 3 or 4 could be in the minority)

$= 10 \times 5 \times 6 \times 3 = 900$

So $P(only\ digits\ 1\ to\ 5\ chosen\ \&\ 3\ different\ digits)$

$= \frac{600+900}{90000} = \frac{1500}{90000}$

To find the number of ways of choosing numbers, using 4 digits from 1 to 5:

eg 23455

5_{C_4} (no. of ways of selecting 4 numbers such as 2,3,4&5)

$\times 5 \times 4\ \times 3$ ($number\ of\ ways\ of\ placing\ the\ numbers$)

$\times 4$ (as each of the 4 numbers could be in the majority)

$= 5 \times 60 \times 4 = 1200$

So $P(only\ digits\ 1\ to\ 5\ chosen\ \&\ 4\ different\ digits)$

$= \frac{1200}{90000}$

To find the number of ways of choosing numbers, using 5 digits from 1 to 5:

eg 23451

number of ways $= 5! = 120$

So $P(only\ digits\ 1\ to\ 5\ chosen\ \&\ 5\ different\ digits)$

$= \frac{120}{90000}$

Thus expected number of different digits

$= \frac{1}{90000\left(\frac{5}{9(16)}\right)}((1)(5) + (2)(300) + (3)(1500) + (4)(1200) + (5)(120))$

$= \frac{16}{10000}(1 + 120 + 900 + 960 + 120) = \frac{16(2101)}{10000} = \frac{33616}{10000} = 3.3616$

as required