STEP 2011, Paper 3, Q6 – Solution (2 pages; 12/6/18)

To get from T to V, Parts looks promising:

$$\int_{1/3}^{1/2} \frac{artanht}{t}\, dt = [artanht.\, lnt]_{\frac{1}{3}}^{\frac{1}{2}} - \int_{1/3}^{1/2} \frac{lnt}{1-t^2}\, dt$$

From the formulae booklet,

$$[artanht.\, lnt]_{\frac{1}{3}}^{\frac{1}{2}} = [\tfrac{1}{2} ln\left(\frac{1+t}{1-t}\right) lnt]_{1/3}^{1/2} \quad \text{(as |t|<1)}$$

$$= \frac{1}{2}\{ln\left(\frac{\frac{3}{2}}{\frac{1}{2}}\right) ln\left(\frac{1}{2}\right) - ln\left(\frac{\frac{4}{3}}{\frac{2}{3}}\right) ln\left(\frac{1}{3}\right)\}$$

$$= -1/2\{ln3ln2 - ln2ln3\} = 0$$

So $\int_{1/3}^{1/2} \frac{artanht}{t}\, dt = -\int_{1/3}^{1/2} \frac{lnt}{1-t^2}\, dt$, and thus $T = V$

For U, we can try using the definition of $sinhu$ as $\frac{1}{2}(e^u - e^{-u})$, with a view to making the substitution $z = e^u$, so that the limits $ln2$ & $ln3$ are converted to 2 & 3; a further reciprocal substitution may then lead us to V (it isn't clear what will happen to the integrand itself, but the presence of exponential and log functions between U & V is encouraging).

So $U = \int_{ln2}^{ln3} \frac{u}{e^u - e^{-u}} du = \int_{ln2}^{ln3} \frac{ue^u}{e^{2u}-1} du$

Let $z = e^u$, so that $dz = e^u du$

and $U = \int_2^3 \frac{lnz}{z^2-1} dz$

Then let $v = 1/z$, so that $dv = -\frac{1}{z^2} dz$

and $U = \int_{1/2}^{1/3} \frac{-lnv}{\left(\frac{1}{v^2}-1\right)}(-\frac{1}{v^2}dv)$

$= \int_{1/2}^{1/3} \frac{lnv}{1-v^2}dv = -\int_{1/3}^{1/2} \frac{lnv}{1-v^2}dv = V$

For X, Parts could work, integrating 1 to give x (to become the u in U), and noting that $\frac{d}{dx}ln(cothx) = \frac{1}{cothx}\frac{d}{dx}(cothx)$, which may (if we're lucky) lead us to the $sinhu$; bearing in mind that the limits of U & X are only related by $u = 2x$

Thus $\frac{1}{cothx}\frac{d}{dx}(cothx) = tanhx\frac{d}{dx}\left(\frac{1}{tanhx}\right)$

$= tanhx(-1)(tanhx)^{-2}sech^2x = -\frac{coshx}{sinhx(coshx)^2} = -\frac{2}{sinh(2x)}$

So, applying Parts, $X = [xln(cothx)]_{\frac{1}{2}ln2}^{\frac{1}{2}ln3} - \int_{\frac{1}{2}ln2}^{\frac{1}{2}ln3} \frac{-2x}{sinh(2x)}dx$

Now, $cothx = \frac{e^x+e^{-x}}{e^x-e^{-x}} = \frac{e^{2x}+1}{e^{2x}-1}$,

so that $coth\left(\frac{1}{2}ln3\right) = \frac{3+1}{3-1} = 2$ & $coth\left(\frac{1}{2}ln2\right) = \frac{2+1}{2-1} = 3$

and thus, with $u = 2x$, so that $du = 2dx$,

$X = \left\{\frac{1}{2}ln3ln2 - \frac{1}{2}ln2ln3\right\} + \int_{ln2}^{ln3} \frac{u}{2sinhu}du = U$

Thus we have shown that $T = V, U = V$ & $X = U$, and so the 4 integrals are equal.