STEP 2011, Paper 3, Q4 – Solution (2 pages; 12/6/18)

(i) The 1st term on the RHS of (*) is the area (1) in Fig. 1, and the 2nd term is the area (2), since if we consider strips parallel to the x-axis: they will have area $x\delta y$, and $y = f(x) \Rightarrow x = f^{-1}(y)$, giving $\int_0^b f^{-1}(y)dy$, as the strips become infinitesimally narrow.

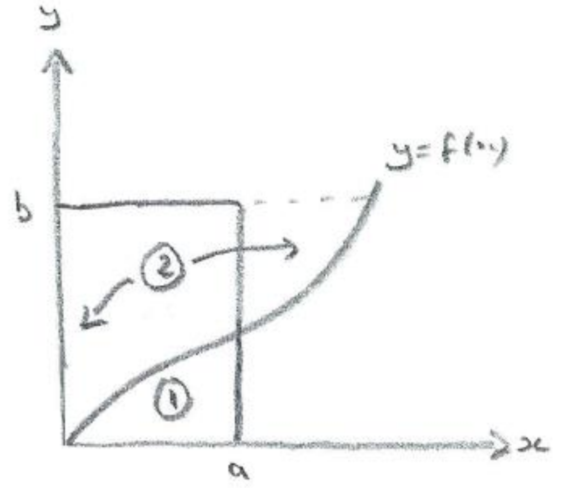

Fig. 1

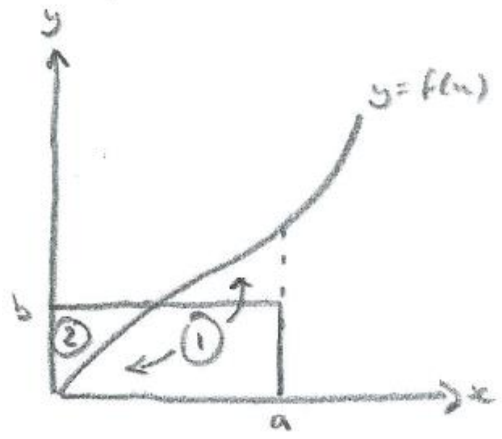

Fig. 2

Then, from Figs 1 & 2 (depending on whether $b > f(a)$ *or* $b < f(a)$), it can be seen that $(1) + (2) > ab$, whilst $(1) + (2) = ab$ when $b = f(a)$.

(ii) $y = x^{p-1} \Rightarrow x = y^{(\frac{1}{p-1})}$, so that $f^{-1}(y) = y^{(\frac{1}{p-1})}$

As $f(x) = x^{p-1}$ is a continuous, increasing function (ie has positive gradient) for $p > 1$, and $f(0) = 0$,

$(*) \Rightarrow ab \leq \int_0^a x^{p-1}dx + \int_0^b y^{\left(\frac{1}{p-1}\right)}dy = \left[\frac{1}{p}x^p\right]\begin{matrix}a\\0\end{matrix} + \left[\frac{y^{\left(\frac{1}{p-1}\right)+1}}{\left(\frac{1}{p-1}\right)+1}\right]\begin{matrix}b\\0\end{matrix}$

$= \frac{a^p}{p} + \frac{b^{\frac{1+p-1}{p-1}}}{\frac{1+p-1}{p-1}}$

Then, as $\frac{p-1}{1+p-1} = \frac{p-1}{p} = 1 - \frac{1}{p} = \frac{1}{q}$, $\frac{1+p-1}{p-1} = q$

and $\frac{a^p}{p} + \frac{b^{\frac{1+p-1}{p-1}}}{\frac{1+p-1}{p-1}} = \frac{a^p}{p} + \frac{b^q}{q}$

so that $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$, as required.

$b = f(a) \Rightarrow b = a^{p-1}$,

so that $ab = a^p$ and $\frac{a^p}{p} + \frac{b^q}{q} = \frac{a^p}{p} + \frac{a^{(p-1)q}}{q}$

$= a^p(\frac{1}{p} + \frac{a^{pq-q-p}}{q})$

As $\frac{1}{p} + \frac{1}{q} = 1, \frac{q+p}{pq} = 1\ ; q + p = pq; pq - q - p = 0$

and hence $\frac{a^p}{p} + \frac{b^q}{q} = a^p\left(\frac{1}{p} + \frac{1}{q}\right) = a^p = ab$, as required.

(iii) Let $f(x) = sinx$

As $f(x)$ is a continuous, increasing function in the interval $[0, \frac{\pi}{2}]$ and $f(0) = 0$,

$(*) \Rightarrow ab \leq \int_0^a sinxdx + \int_0^b arcsiny\, dy$

$= -cosa + 1 + [yarcsiny]_0^b - \int_0^b \frac{y}{\sqrt{1-y^2}} dy$ (by Parts)

$= -cosa + 1 + barcsinb + \frac{1}{2}\int_0^b \frac{-2y}{\sqrt{1-y^2}} dy$

$= -cosa + 1 + barcsinb + \frac{1}{2}\left[\frac{\sqrt{1-y^2}}{1/2}\right]_0^b$

$= -cosa + 1 + barcsinb + \sqrt{1-b^2} - 1$

$= barcsinb + \sqrt{1-b^2} - cosa$

Then, with $b = t^{-1}$, so that $t \geq 1$, and multiplying by t:

$a \leq \arcsin(t^{-1}) + t\sqrt{1 - \frac{1}{t^2}} - tcosa$

so that $\arcsin(t^{-1}) \geq a + tcosa - \sqrt{t^2 - 1}$

and setting $a = 0$, $\arcsin(t^{-1}) \geq t - \sqrt{t^2 - 1}$, as required.