STEP 2011, Paper 3, Q3 – Solution (3 pages; 12/6/18)

Suppose that $x^3 - 3px + q = a(x-\alpha)^3 + b(x-\beta)^3$ (1)

[Note that $q^2 \neq 4p^3$ is the condition for the discriminant of

$pt^2 - qt + p^2 = 0$ not to be zero; ie for $\alpha \neq \beta$ (if $\alpha = \beta$, then the desired form would reduce to $(a+b)(x-\alpha)^3$, which wouldn't be possible, due to the presence of an x^2 term on only one side of (1)).]

Equating coefficients:

x^3: $1 = a + b$ (2)

x^2: $0 = -3a\alpha - 3b\beta \Rightarrow a\alpha + b\beta = 0$ (3)

x: $-3p = 3a\alpha^2 + 3b\beta^2 \Rightarrow p = -(a\alpha^2 + b\beta^2)$ (4)

x^0: $q = -a\alpha^3 - b\beta^3$ (5)

α & β will be roots of $pt^2 - qt + p^2 = 0$ if and only if $\alpha + \beta = \frac{q}{p}$ and $\alpha\beta = \frac{p^2}{p} = p$

From (4) & (5), $\frac{q}{p} = \frac{-a\alpha^3 - b\beta^3}{-(a\alpha^2 + b\beta^2)}$ (6)

From (2), $b = 1 - a$, and from (3), $\beta^3 = \left(\frac{-a\alpha}{b}\right)^3$ & $\beta^2 = \left(\frac{-a\alpha}{b}\right)^2$

Substituting into (6) gives

$$\frac{q}{p} = \frac{-a\alpha^3 + \frac{a^3\alpha^3}{b^2}}{-(a\alpha^2 + \frac{a^2\alpha^2}{b})} = \frac{b^2\alpha - a^2\alpha}{b^2 + ba} = \frac{[(1-a)^2 - a^2]\alpha}{(1-a)^2 + (1-a)a}$$

$$= \frac{[1-2a]\alpha}{1-a} = \frac{(b-a)\alpha}{b}$$

From (3), $\frac{\alpha}{b} = -\frac{\beta}{a}$, so the above expression becomes

$\alpha - a\left(-\frac{\beta}{a}\right) = \alpha + \beta$, as required

[The above approach of trying to simplify the fraction $\frac{q}{p}$ is slightly risky, in that we could go round in circles. The approach in the official solutions, whereby a & b are first expressed in terms of α & β, and then substituted into (4), to give p in terms of α & β, is perhaps more reliable.]

To show that $p = \alpha\beta$:

From (1) & (3), $b = 1 - a$ & $a\alpha + b\beta = 0$,

so that $a\alpha + (1 - a)\beta = 0$

and hence $a(\alpha - \beta) = -\beta$, giving $a = \frac{\beta}{\beta - \alpha}$

and $b = 1 - \frac{\beta}{\beta - \alpha} = \frac{-\alpha}{\beta - \alpha} = \frac{\alpha}{\alpha - \beta}$

Then, from (4), $p = -(a\alpha^2 + b\beta^2) = -\frac{1}{\beta - \alpha}(\beta\alpha^2 - \alpha\beta^2)$

$= \frac{\alpha\beta(\alpha - \beta)}{\alpha - \beta} = \alpha\beta$, as required

With $p = 8$ & $q = 48$, the quadratic equation becomes

$8t^2 - 48t + 64 = 0$

or $t^2 - 6t + 8 = 0$, so that $(t - 4)(t - 2) = 0$ and $\alpha = 4, \beta = 2$ (or the other way round)

Then $a = \frac{2}{-2} = -1$ & $b = \frac{4}{2} = 2$

and $x^3 - 24x + 49 = 0$ can then be written as

$-(x - 4)^3 + 2(x - 2)^3 = 0$

so that $2 = \left(\frac{x-4}{x-2}\right)^3$ and $\frac{x-4}{x-2} = 2^{\frac{1}{3}}\lambda$, where λ is one of the cube roots of 1; ie $1, \omega\ or\ \omega^2$

Then $x - 4 = 2^{\frac{1}{3}}\lambda(x-2)$,

and $x\left(1 - 2^{\frac{1}{3}}\lambda\right) = 4 - 2\left(2^{\frac{1}{3}}\lambda\right)$,

so that $x = \frac{2(2-2^{\frac{1}{3}}\lambda)}{1-2^{\frac{1}{3}}\lambda}$

[When $p = r^2$ & $q = 2r^3$, the above method breaks down, as mentioned at the start.]

By the Factor theorem, $x - r$ is seen to be a factor of

$x^3 - 3r^2x + 2r^3$

and $x^3 - 3r^2x + 2r^3 = (x-r)(x^2 + rx - 2r^2)$

$= (x-r)(x+2r)(x-r)$

so that the roots are $x = r$ (repeated) and $x = -2r$

[The last past is rather easier than might have been expected!]