STEP 2011, Paper 2, Q11 – Solution (3 pages; 12/6/18)

(i) WLOG [without loss of generality] let $|\overrightarrow{PB}| = 1$

Then $\overrightarrow{PB} = cos(BPO)\underline{k} + sin(BPO)\underline{\widehat{OB}}$

where $\underline{\widehat{OB}}$ is a unit vector in the direction of $\overrightarrow{OB}$

$= cos(90° - \theta)\underline{k} + sin(90° - \theta)(cos(90° + \theta)\underline{j} - \sin(90° + \theta)\,\underline{i})$

$= sin\theta\underline{k} + cos\theta(-sin\theta\underline{j} - cos\theta\underline{i})$

[noting that $cos(90° + \theta)$ is $cos\theta$ translated 90° to the left, and similarly for $\sin(90° + \theta)$]

$= -\left(\frac{1}{\sqrt{3}}\right)^2 \underline{i} - \left(\frac{1}{\sqrt{3}}\right)\left(\frac{\sqrt{2}}{\sqrt{3}}\right)\underline{j} + \frac{\sqrt{2}}{\sqrt{3}}\underline{k}$

[From $tan\theta = \sqrt{2}$, we can create the Pythagorean triple $1, \sqrt{2}, \sqrt{3}$]

$= -\frac{1}{3}\underline{i} - \frac{\sqrt{2}}{3}\underline{j} + \frac{\sqrt{2}}{\sqrt{3}}\underline{k}$

(ii) Similarly, taking $|\overrightarrow{PA}| = 1$,

$\overrightarrow{PA} = cos(APO)\underline{k} + sin(\text{APO})\underline{j}$

$= cos(30°)\underline{k} + sin(30°)\underline{j}$

$= \frac{1}{2}\underline{j} + \frac{\sqrt{3}}{2}\underline{k}$

And taking $|\overrightarrow{PC}| = 1$,

$\overrightarrow{PC} = cos(CPO)\underline{k} + sin(\text{CPO})\underline{\widehat{OC}}$

where $\underline{\widehat{OC}}$ is a unit vector in the direction of $\overrightarrow{OC}$

$= cos(60°)\underline{k} + sin(60°)(cos\phi\underline{i} - sin\phi\underline{j})$

$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{4}{3\sqrt{2}}\right)\underline{i} - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{3}\right)\underline{j} + \frac{1}{2}\underline{k}$

$= \left(\frac{\sqrt{6}}{3}\right)\underline{i} - \left(\frac{\sqrt{3}}{6}\right)\underline{j} + \frac{1}{2}\underline{k}$

Then the forces acting on P are:

$-W\underline{k}$

U$(-\frac{1}{3}\underline{i} - \frac{\sqrt{2}}{3}\underline{j} + \frac{\sqrt{2}}{\sqrt{3}}\underline{k})$

$T(\frac{1}{2}\underline{j} + \frac{\sqrt{3}}{2}\underline{k})$

and $V(\left(\frac{\sqrt{6}}{3}\right)\underline{i} - \left(\frac{\sqrt{3}}{6}\right)\underline{j} + \frac{1}{2}\underline{k})$

(iii) As P is in equilibrium, the net force on it is zero.

Resolving in the $\underline{i}$ direction,

$-\frac{U}{3} + \frac{V\sqrt{6}}{3} = 0 \Rightarrow U = \sqrt{6}V$, as required.

Resolving in the $\underline{j}$ direction,

$-\frac{U\sqrt{2}}{3} + \frac{T}{2} - \frac{V\sqrt{3}}{6} = 0$

so that $3T = 2U\sqrt{2} + V\sqrt{3} = V\left(2\sqrt{12} + \sqrt{3}\right) = 5\sqrt{3}V$

Resolving in the $\underline{k}$ direction,

$-W + \frac{U\sqrt{2}}{\sqrt{3}} + \frac{T\sqrt{3}}{2} - \frac{V}{2} = 0$

so that $-W + \frac{V\sqrt{12}}{\sqrt{3}} + \frac{5V}{2} + \frac{V}{2} = 0$

and $-W + 5V = 0$

Thus $V = \frac{W}{5}$, $U = \frac{\sqrt{6}W}{5}$, $T = \frac{5\sqrt{3}W}{15} = \frac{\sqrt{3}W}{3}$