STEP 2011, Paper 1, Q6 – Solution (2 pages; 11/6/18)

[The main catch with this question is the need to demonstrate that the results hold for $r = 0$ and 1 (depending on the method used).

For part (ii), the standard dilemma is whether the result of part (i) is to be used, or the method of part (i). Unusually, either approach is possible in this question (as described in the official solutions); though using the result is easier.]

$(1-x)^{-3} = 1 + (-3)(-x) + \frac{(-3)(-4)}{2!}(-x)^2 + \cdots$

$+ \frac{(-3)(-4)\ldots[r\ terms]}{r!}(-x)^r + \cdots$

The coeff. of x^r is $\frac{(r+2)!/2}{r!} = \frac{1}{2}(r+1)(r+2)$, as required.

(i) Expanding

$(1 - x + 2x^2)(1 + (-3)(-x) + \frac{(-3)(-4)}{2!}(-x)^2 + \cdots),$

the coeff. of x^r is $\frac{1}{2}(r+1)(r+2) - \frac{1}{2}([r-1]+1)([r-1]+2)$

$+(2)\frac{1}{2}([r-2]+1)([r-2]+2)$

provided that $r - 2 \geq 0$

$= \frac{1}{2}\{r^2 + 3r + 2 - r(r+1) + 2(r-1)r\}$

$= \frac{1}{2}\{2r^2 + 2\} = r^2 + 1$

When $r = 1$, the coeff. of x^r is $3 - 1 = r^2 + 1$,

and when $r = 0$, the coeff. of x^r is $1 = r^2 + 1$

So the coeff. of x^r is $r^2 + 1$ for $r \geq 0$

The sum of the given series is $\sum_{r=0}^{\infty}\frac{r^2+1}{2^r} = \sum_{r=0}^{\infty}(r^2+1)x^r$ with $x = \frac{1}{2}$

$= \frac{1-x+2x^2}{(1-x)^3}$ with $x = \frac{1}{2}$

$= \frac{1-\frac{1}{2}+\frac{1}{2}}{\frac{1}{8}} = 8$

(ii) $1 + 2 + \frac{9}{4} + 2 + \frac{25}{16} + \frac{9}{8} + \frac{49}{64} + \cdots$

$= 1 + \frac{4}{2} + \frac{9}{4} + \frac{16}{8} + \frac{25}{16} + \frac{36}{32} + \frac{49}{64} + \cdots$

$= \sum_{r=0}^{\infty}\frac{(r+1)^2}{2^r} = \sum_{u=1}^{\infty}\frac{u^2}{2^{u-1}}$

$= \sum_{u=0}^{\infty}\frac{u^2}{2^{u-1}} = 2\sum_{u=0}^{\infty}\frac{u^2}{2^u}$

$= 2\sum_{u=0}^{\infty}\frac{u^2+1}{2^u} - 2\sum_{u=0}^{\infty}\frac{1}{2^u}$

$= 2(8) - \frac{2}{1-\frac{1}{2}} = 16 - 4 = 12$