STEP 2011, Paper 1, Q1 – Solution (2 pages; 11/6/18)

(i) $\frac{a}{x}+\frac{b}{y}=1 \Rightarrow -\frac{a}{x^2}-\frac{b}{y^2}\frac{dy}{dx}=0$

$b \neq 0 \Rightarrow \frac{dy}{dx}=-\left(\frac{a}{x^2}\right) \div \left(\frac{b}{y^2}\right)=-\frac{ay^2}{bx^2}$, as required

As (p,q) lies on the line and the curve,

$ap+bq=1$ (1) & $\frac{a}{p}+\frac{b}{q}=1$ (2)

Equal gradients $\Rightarrow \frac{-a}{b}=-\frac{aq^2}{bp^2} \Rightarrow p^2=q^2 \Rightarrow p=\pm q$, as required.

From (1) & (2), $p=q \Rightarrow p(a+b)=1$ & $\frac{1}{p}(a+b)=1$

Multiplying these results together: $(a+b)^2=1$

Alternatively, $p=-q \Rightarrow p(a-b)=1$ & $\frac{1}{p}(a-b)=1$,

giving $(a-b)^2=1$

(ii) $\frac{a}{x}-\frac{b}{y}=1 \Rightarrow -\frac{a}{x^2}+\frac{b}{y^2}\frac{dy}{dx}=0$

$\Rightarrow \frac{dy}{dx}=\left(\frac{a}{x^2}\right) \div \left(\frac{b}{y^2}\right)=\frac{ay^2}{bx^2}$ (as $ab \neq 0 \Rightarrow b \neq 0$)

Let (p,q) be the point at which the line $ax+by=1$ is a normal to the curve.

As $ab \neq 0 \Rightarrow a \neq 0$ also,

$\frac{-a}{b}=-\frac{1}{\left(\frac{aq^2}{bp^2}\right)} \Rightarrow a^2q^2=b^2p^2 \Rightarrow aq=\pm bp$

[The official sol'ns don't require $a \neq 0$ (and don't refer to the fact that $ab \neq 0$), but are implicitly relying on the fact that the normal exists (otherwise $\left(\frac{aq^2}{bp^2}\right)\left(\frac{-a}{b}\right)=-1$ is not possible). Instead they mention that $pq \neq 0$, without any justification (perhaps they were confusing it with $ab \neq 0$!) - though neither p or q could be zero, given that (p, q) lies on $\frac{a}{x} - \frac{b}{y} = 1$]

As (p, q) lies on the line and the curve,

$ap + bq = 1$ (3) & $\frac{a}{p} - \frac{b}{q} = 1$ (4)

Then if $aq = bp$, (3) $\Rightarrow \text{ap} + \text{b}\left(\frac{bp}{a}\right) = 1 \Rightarrow \text{p}\left(\text{a} + \frac{b^2}{a}\right) = 1$

and (4) $\Rightarrow \frac{a}{p} - \frac{b}{\left(\frac{bp}{a}\right)} = 1 \Rightarrow \frac{1}{p}(\text{a} - \text{a}) = 1$, which is impossible

If instead $aq = -bp$,

then (3) $\Rightarrow \text{ap} + \text{b}\left(\frac{-bp}{a}\right) = 1 \Rightarrow \text{p}\left(\text{a} - \frac{b^2}{a}\right) = 1$

and (4) $\Rightarrow \frac{a}{p} - \frac{b}{\left(-\frac{bp}{a}\right)} = 1 \Rightarrow \frac{1}{p}(\text{a} + \text{a}) = 1$

Multiplying these results together gives

$\left(\text{a} - \frac{b^2}{a}\right)(2a) = 1 \Rightarrow 2a^2 - 2b^2 = 1 \Rightarrow a^2 - b^2 = \frac{1}{2}$,

as required.