STEP 2010, Paper 3, Q8 – Solution (4 pages; 10/6/18)

We can write $\frac{dI}{dx} = \frac{P(x)}{(Q(x))^2}$ (where I is the integral to be found). It is then clear that we need to be applying the quotient rule to

$I = \frac{R(x)}{Q(x)}$

(i) Let $P(x) = 5x^2 - 4x - 3$, $Q(x) = 1 + 2x + 3x^2$,

$R(x) = a + bx + cx^2$

Then suppose that $P(x) = Q(x)R'(x) - Q'(x)R(x)$

$= (1 + 2x + 3x^2)(b + 2cx) - (2 + 6x)(a + bx + cx^2)$

Equating this to $5x^2 - 4x - 3$ gives

$x^2: 5 = 3b + 4c - 2c^2 - 6b \Rightarrow 5 = 4c - 2c^2 - 3b$ (1)

$x: -4 = 2b + 2c - 2b - 6a \Rightarrow 2c - 6a = -4 \Rightarrow c = 3a - 2$ (2)

$x^0: -3 = b - 2a \Rightarrow b = 2a - 3$ (3)

Substituting for c & b into (1), from (2) & (3):

$5 = 4(3a - 2) - 2(3a - 2)^2 - 3(2a - 3)$

$\Rightarrow 0 = -18a^2 + a(12 + 24 - 6) - 8 - 8 + 9 - 5$

$\Rightarrow 18a^2 - 30a + 12 = 0$

$\Rightarrow 3a^2 - 5a + 2 = 0$

$\Rightarrow (3a - 2)(a - 1) = 0$

$\Rightarrow a = 1 \text{ or } \frac{2}{3}$

The two possible expressions for $R(x)$ should give rise to answers for the integral that differ only by a constant.

When $a = 1, b = -1$ & $c = 1$ [from (2) & (3)]. One expression for the answer is then $\frac{R(x)}{Q(x)} = \frac{1-x+x^2}{1+2x+3x^2} + C$

When $a = \frac{2}{3}, b = -\frac{5}{3}$ & $c = 0$, so that $\frac{R(x)}{Q(x)} = \frac{2/3-5x/3}{1+2x+3x^2}$

To confirm that the two expressions for $\frac{R(x)}{Q(x)}$ only differ by a constant: $\frac{1-x+x^2}{1+2x+3x^2} - \frac{\frac{2}{3}-\frac{5x}{3}}{1+2x+3x^2} = \frac{3(1-x+x^2)-(2-5x)}{3(1+2x+3x^2)}$

$= \frac{1+2x+3x^2}{3(1+2x+3x^2)} = \frac{1}{3}$

(ii) After dividing both sides of the equation by $1 + cosx + 2sinx$, the integrating factor is

$\exp\{\int \frac{sinx-2cosx}{1+cosx+2sinx}\, dx\} = \exp\{-\ln|1 + cosx + 2sinx|\}$

$= \frac{1}{1+cosx+2sinx}$

Multiplying by the integrating factor gives:

$\frac{d}{dx}\left(\frac{y}{1+cosx+2sinx}\right) = \frac{5-3cosx+4sinx}{(1+cosx+2sinx)^2}$,

so that $\frac{y}{1+cosx+2sinx} = \int \frac{5-3cosx+4sinx}{(1+cosx+2sinx)^2}\, dx$

Let $P(x) = 5 - 3cosx + 4sinx$, $Q(x) = 1 + cosx + 2sinx$

and $R(x) = a + bcosx + csinx$, and suppose that

$P(x) = Q(x)R'(x) - Q'(x)R(x)$

[we can't be sure that this version of the method in (i) will work, as there is the difference that $(x)(x) = x^2$; whereas $cosx\ cosx \neq sinx$]

Then $5 - 3cosx + 4sinx = (1 + cosx + 2sinx)(-bsinx + ccosx)$

$-(a + bcosx + csinx)(-sinx + 2cosx)$

$= (-bsinx + ccosx) + (-bcosxsinx + ccos^2x)$

$+(-2bsin^2x + 2csinxcosx) + (asinx - 2acosx)$

$+(bcosxsinx - 2bcos^2x)$

$+(csin^2x - 2csinxcosx)$

$= c - 2b + cosx(c - 2a) + sinx(-b + a)$

Equating the coefficients of $cosx, sinx$ & the constant term:

$5 = c - 2b$ (1)

$-3 = c - 2a$ (2)

$4 = -b + a$ (3)

Then $(1) - (2) \Rightarrow 8 = 2a - 2b \Rightarrow 4 = a - b$ (duplicating (3))

[Thus, had the coefficient of $sinx$ on the RHS of the original equation not been 4, for example, the above approach wouldn't work.]

We can therefore choose any value for one of a, b or c.

For example, let $a = 0$, so that $b = -4$ & $c = -3$

so that $\frac{y}{1+cosx+2sinx} = \frac{R(x)}{Q(x)} + C = \frac{-4cosx-3sinx}{1+cosx+2sinx} + C$

and $y = -4cosx - 3sinx + C(1 + cosx + 2sinx)$

[Other possible values for a, b & c should lead to a solution obtainable by choosing a suitable value for C;

eg $b = 0 \Rightarrow c = 5$ & $a = 4$;

giving $y = 4 + 5sinx + C'(1 + cosx + 2sinx)$

Equating the two solutions then gives

$-4cosx - 3sinx + C(1 + cosx + 2sinx)$

$= 4 + 5sinx + C'(1 + cosx + 2sinx)$

Equating coefficients then gives:

constant: $C = 4 + C'$ (1)

$cosx$: $-4 + C = C'$ (2)

$sinx$: $-3 + 2C = 5 + 2C'$ (3)

Thus (1) & (2) are consistent, and (3) gives

$2C = 8 + 2C'$, so that $C = 4 + C'$,

and hence all 3 equations are consistent, and thus a suitable value

$(4 + C')$ can be found for C]