STEP 2010, Paper 2, Q7 – Solution (2 pages; 9/6/18)

(i) $y = x^3 - 3qx - q(1+q)$ (1)

$y' = 3x^2 - 3q$

$y' = 0 \Rightarrow x = \pm\sqrt{q}$

As the coefficient of x^3 in (1) is positive, there must be a maximum at $x = -\sqrt{q}$ and a minimum at $x = \sqrt{q}$

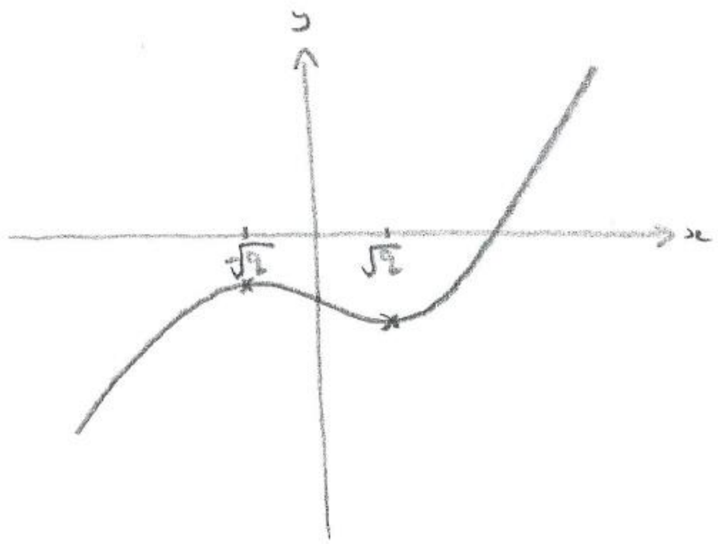
$x = \sqrt{q} \Rightarrow y = q\sqrt{q} - 3q\sqrt{q} - q - q^2$

$= -(2q\sqrt{q} + q + q^2) < 0$, as $q > 0$

$x = -\sqrt{q} \Rightarrow y = -q\sqrt{q} + 3q\sqrt{q} - q - q^2$

$= -q(-2\sqrt{q} + 1 + q) = -q(1-\sqrt{q})^2 < 0$, as $q > 0$ and $q \neq 1$

Thus both the maximum and the minimum lie below the x-axis, and so the curve crosses the x-axis only once (when $x > \sqrt{q}$), as shown below.



(ii) Substituting $x = u + q/u$ into (1) gives

$(u + q/u)^3 - 3q\left(u + \frac{q}{u}\right) - q(1+q) = 0$

$\Rightarrow u^3 + 3uq + \frac{3q^2}{u} + \frac{q^3}{u^3} - 3qu - \frac{3q^2}{u} - q - q^2 = 0$

$\Rightarrow u^3 + \frac{q^3}{u^3} - q - q^2 = 0$

$\Rightarrow (u^3)^2 - q(1+q)u^3 + q^3 = 0$

$\Rightarrow u^3 = \frac{q(1+q) \pm \sqrt{q^2(1+q)^2 - 4q^3}}{2} = \frac{q(1+q) \pm \sqrt{q^2(1-q)^2}}{2}$

$= \frac{q(1+q) \pm q(1-q)}{2} = q \;\; or \; q^2$

$\Rightarrow u = q^{1/3} \;\; or \;\; q^{2/3}$

$\Rightarrow x = q^{1/3} + q^{2/3}$ (or $q^{2/3} + q^{1/3}$)

(iii) $(\alpha + \beta)^3 = \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3$,

so that $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$,

and hence if $t^2 - pt + q = 0$ has roots α & β, then

$\alpha^3 + \beta^3 = p^3 - 3qp$, as required (2)

If either $\alpha = \beta^2$ or $\beta = \alpha^2$, then $(\alpha^2 - \beta)(\beta^2 - \alpha) = 0$,

so that $\alpha^2\beta^2 - \alpha^3 - \beta^3 + \alpha\beta = 0$

and hence $q^2 - (p^3 - 3qp) + q = 0$, from (2)

$\Rightarrow p^3 - 3qp - q(1+q) = 0$

$\Rightarrow p = q^{1/3} + q^{2/3}$, from (ii)