STEP 2010, Paper 2, Q5 – Solution (2 pages; 9/6/18)

$cos(2\alpha) = \frac{\overrightarrow{OA}.\overrightarrow{OB}}{|\overrightarrow{OA}||\overrightarrow{OB}|} = \frac{5-1-1}{\sqrt{3}\sqrt{27}} = \frac{3}{9} = \frac{1}{3}$

(i) Let $\overrightarrow{OC} = m\underline{i} + n\underline{j} + p\underline{k}$

Thus we require $\overrightarrow{OC}$ to be inclined equally to $\overrightarrow{OA}$ and $\overrightarrow{OB}$

and hence $\frac{\overrightarrow{OA}.\overrightarrow{OC}}{|\overrightarrow{OA}||\overrightarrow{OC}|} = \frac{\overrightarrow{OB}.\overrightarrow{OC}}{|\overrightarrow{OB}||\overrightarrow{OC}|}$

$\Rightarrow \frac{m+n+p}{\sqrt{3}} = \frac{5m-n-p}{\sqrt{27}}$

$\Rightarrow 3(m+n+p) = 5m-n-p$

$\Rightarrow 2m-4n-4p = 0$

$\Rightarrow m = 2(n+p) \quad (1)$

For $\overrightarrow{OC}$ (and hence L_1) to be the angle bisector of $\angle AOB$, we also require $cos\alpha = \frac{\overrightarrow{OA}.\overrightarrow{OC}}{|\overrightarrow{OA}||\overrightarrow{OC}|} = \frac{m+n+p}{\sqrt{3}\sqrt{m^2+n^2+p^2}}$;

and since $\frac{1}{3} = cos(2\alpha) = cos^2\alpha - sin^2\alpha = 2cos^2\alpha - 1$,

$cos^2\alpha = \frac{2}{3}$, so that $\frac{2}{3} = \frac{(m+n+p)^2}{3(m^2+n^2+p^2)}$

$\Rightarrow 2(m^2+n^2+p^2) = m^2+n^2+p^2+2(mn+mp+np)$

$\Rightarrow m^2+n^2+p^2 = 2(mn+mp+np)$

Then, from (1): $4(n+p)^2+n^2+p^2 = 2(n+p).2(n+p)+2np$

$\Rightarrow n^2+p^2-2np = 0$

$\Rightarrow (n-p)^2 = 0 \Rightarrow n = p$

Thus a possible solution is $n = p = 1, m = 4$

(Any multiple of these values will also provide a suitable direction vector for L_1.)

[Alternatively, in addition to (1), we require $\overrightarrow{OC}$ to be in the plane of $\overrightarrow{OA}$ & $\overrightarrow{OB}$, and hence $\begin{vmatrix} m & 1 & 5 \\ n & 1 & -1 \\ p & 1 & -1 \end{vmatrix} = 0$

(Interpretation 1: $\overrightarrow{OA}, \overrightarrow{OB}$ & $\overrightarrow{OC}$ are linearly dependent;

Interpretation 2: $\overrightarrow{OC}$ is perpendicular to : $\overrightarrow{OA} \times \overrightarrow{OB}$;

Interpretation 3: The volume of the parallelepiped with sides $\overrightarrow{OA}, \overrightarrow{OB}$ & $\overrightarrow{OC}$ is zero.)

So $m(0) - n(-6) + p(-6) = 0$

and hence $n = p$]

(ii) Let $\overrightarrow{OD} = u\underline{i} + v\underline{j} + w\underline{k}$

Then $\frac{\overrightarrow{OA}.\overrightarrow{OD}}{|\overrightarrow{OA}||\overrightarrow{OD}|} = cos\alpha$, so that $\frac{u+v+w}{\sqrt{3}\sqrt{u^2+v^2+w^2}} = \sqrt{\frac{2}{3}}$, from (i)

$\Rightarrow (u + v + w)^2 = 2(u^2 + v^2 + w^2)$

$\Rightarrow 2(uv + uw + vw) = u^2 + v^2 + w^2$

The given surface is therefore a double cone [ie a cone, together with its mirror image in the vertex] with vertex O, centred on $\overrightarrow{OA}$.