STEP 2010, Paper 1, Q5 – Solution (2 pages; 8/6/18)

(i) $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$ (A)

Setting $x = 1$ gives $2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$, as required.

(ii) Differentating both sides of (A) wrt x gives

$n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 \ldots + n\binom{n}{n}x^{n-1}$ (B)

and setting $x = 1$ gives

$n2^{n-1} = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n}$, as required.

(iii) [With integration there is always the issue of the arbitrary constant. Two functions that differ by a constant will differentiate to give the same function, and reversing the process by integration requires the addition of the arbitrary constant. However, we can side-step this issue by using definite integration.]

From (A), $\int_0^1 (1+x)^n dx = \left[\frac{1}{n+1}(1+x)^{n+1}\right]_0^1 = \frac{1}{n+1}(2^{n+1} - 1)$,

whilst $\int_0^1 \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n \, dx$

$= [\binom{n}{0}x + \frac{1}{2}\binom{n}{1}x^2 + \frac{1}{3}\binom{n}{2}x^3 + \cdots + \frac{1}{n+1}\binom{n}{n}x^{n+1}]_0^1$

$= \binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \cdots + \frac{1}{n+1}\binom{n}{n}$

and equating these two expressions gives the required result.

(iv) [The 2^{n-2} on the RHS suggests differentiating twice, though we have a coefficient of $n(n+1)$, rather than $n(n-1)$. On the LHS, n^2 could be produced by differentiating, then multiplying by x and differentiating again.]

Starting from (B) above; ie

$$n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 \ldots + n\binom{n}{n}x^{n-1},$$

multiplying both sides by x gives

$$nx(1+x)^{n-1} = \binom{n}{1}x + 2\binom{n}{2}x^2 + 3\binom{n}{3}x^3 \ldots + n\binom{n}{n}x^n,$$

and differentiating both sides wrt x gives

$$n(1+x)^{n-1} + nx(n-1)(1+x)^{n-2}$$

$$= \binom{n}{1} + 2^2\binom{n}{2}x + 3^2\binom{n}{3}x^2 \ldots + n^2\binom{n}{n}x^{n-1},$$

and setting $x = 1$ gives

$$n(2^{n-1}) + n(n-1)2^{n-2} = \binom{n}{1} + 2^2\binom{n}{2} + 3^2\binom{n}{3} + \cdots + n^2\binom{n}{n}$$

and LHS $= n(2^{n-2})(2+n-1) = n(n+1)2^{n-2}$, as required.

[The official solutions also give an approach for (i)-(iii) based on manipulating binomial coefficients.]