STEP 2009, Paper 3, Q7 - Solution (3 pages; 7/6/18)

(i) Proof by induction:

$f_1(x) = \frac{d}{dx}\left(\frac{1}{1+x^2}\right) = -(1+x^2)^{-2}(2x)$

$f_2(x) = \frac{d}{dx}f_1(x) = 2(1+x^2)^{-3}(2x)(2x) - (1+x^2)^{-2}(2)$

Then $(1+x^2)f_2(x) + 2(1+1)xf_1(x) + 1(1+1)f_0(x)$

$= 8(1+x^2)^{-2}x^2 - 2(1+x^2)^{-1} - 8x^2(1+x^2)^{-2} + 2(1+x^2)^{-1}$

$= 0$

Thus the result is true for $n = 1$.

Assume that the result is true for $n = k$,

so that

$(1+x^2)f_{k+1}(x) + 2(k+1)xf_k(x) + k(k+1)f_{k-1}(x) = 0 \quad (1)$

Result to prove:

$(1+x^2)f_{k+2}(x) + 2(k+2)xf_{k+1}(x) + (k+1)(k+2)f_k(x) = 0$

Differentiating (1):

$2xf_{k+1}(x) + (1+x^2)f'_{k+1}(x) + 2(k+1)f_k(x)$

$+2(k+1)xf'_k(x) + k(k+1)f'_{k-1}(x) = 0$

$\Rightarrow 2xf_{k+1}(x) + (1+x^2)f_{k+2}(x) + 2(k+1)f_k(x)$

$+2(k+1)xf_{k+1}(x) + k(k+1)f_k(x) = 0$

$\Rightarrow (1+x^2)f_{k+2}(x) + f_{k+1}(x)\{2x + 2(k+1)x\}$

$+f_k(x)\{2(k+1) + k(k+1)\} = 0 \Rightarrow$

$(1+x^2)f_{k+2}(x) + 2(k+2)xf_{k+1}(x) + (k+1)(k+2)f_k(x) = 0,$

and this is the required result for $n = k+1$

So, if the result is true for $n = k$, then it is true for $n = k + 1$. As it is true for $n = 1$, it is therefore true for $n = 2,3, \dots$, and hence

$$(1 + x^2)f_{n+1}(x) + 2(n + 1)xf_n(x) + n(n + 1)f_{n-1}(x) = 0$$

for all $n \in \mathbb{Z}^+$, by the principle of induction.

(ii) $P_0(x) = (1 + x^2)\left(\frac{1}{1+x^2}\right) = 1$

$P_1(x) = (1 + x^2)^2 \frac{d}{dx} f_0(x) = (1 + x^2)^2(-1)(1 + x^2)^{-2}(2x)$

$= -2x$

$P_2(x) = (1 + x^2)^3 \frac{d}{dx} f_1(x)$

$= (1 + x^2)^3\{2(1 + x^2)^{-3}(2x)(2x) - (1 + x^2)^{-2}(2)\}$

$= 8x^2 - 2(1 + x^2) = 6x^2 - 2$

$P_{n+1}(x) - (1 + x^2) \frac{d}{dx} P_n(x) + 2(n + 1)xP_n(x)$

$= (1 + x^2)^{n+2} f_{n+1}(x)$

$-(1 + x^2)\{(n + 1)(1 + x^2)^n(2x)f_n(x) + (1 + x^2)^{n+1} f_{n+1}(x)\}$

$+2(n + 1)x(1 + x^2)^{n+1} f_n(x)$

$= (1 + x^2)^{n+1}\{(1 + x^2)f_{n+1}(x) - 2(n + 1)xf_n(x)$

$-(1 + x^2)f_{n+1}(x) + 2(n + 1)xf_n(x)\}$

$= 0$, as required (1)

Proof by induction

$P_0(x) = 1$ is a polynomial of degree 0

Assume that $P_k(x)$ is a polynomial of degree k, so that $P_k(x) = \sum_{i=0}^{k} a_i x^i$, where $a_k \neq 0$

Then, from (1),

$P_{k+1}(x) = (1 + x^2)\{\sum_{i=1}^{k} i a_i x^{i-1}\} + 2(k+1)x \sum_{i=0}^{k} a_i x^i$

The highest power is x^{k+1} , and the coefficient is

$k a_k + 2(k+1)a_k$, which is non-zero for $k \geq 0$

So, if $P_k(x)$ is a polynomial of degree k, $P_{k+1}(x)$ will be a polynomial of degree k+1.

Hence, if the result is true for $n = 0$, it is true for $n = 1,2, \ldots$ and hence for all $n \in \mathbb{Z}, n \geq 0$, by the principle of induction.

[The official solutions suggest showing first of all that $P_{k+1}(x)$ is of degree not greater than $k + 1$, and then that there is a term involving x^{k+1}, but it isn't clear why this two stage method is necessary. Also, the Examiners' report mentions that, for the last part, candidates often fell by the wayside; "especially those who attempted it by induction" (almost implying that there is a better method). But the official solution suggests using induction, and this seems to be a perfectly good way of proceeding.]