STEP 2009, Paper 2, Q13 – Solution (2 pages; 5/6/18)

E(cost for A)$= P(1 \text{ failed engine}) \times K$

$+[1 - P(0 \text{ failed engines}) - P(1 \text{ failed engine})] \times 4K$

$= 4pq^3K + [1 - 4pq^3 - q^4](4K)$

[In order to be able to create a factor p, we could expand

$1 - q^4 = 1 - (1-p)^4$, in order to get rid of the 1; though there is a slightly quicker method]

Method 1

$= 4Kp(q^3 - 4q^3) + (4K)\{1 - (1-p)^4\}$

$= 4Kp(-3q^3) + (4K)\{4p - 6p^2 + 4p^3 - p^4\}$

$= 4Kp(-3q^3 + 4 - 6p + 4p^2 - p^3)$

$= 4Kp\{(-3q^3 + 4 - 6(1-q) + 4(1-q)^2 - (1-q)^3\}$

$= 4Kp\{(4 - 6 + 4 - 1) + q(6 - 8 + 3) + q^2(4 - 3) + q^3(-3 + 1)\}$

$= 4Kp\{1 + q + q^2 - 2q^3\}$, as required

Method 2

[Having spotted that we can get rid of the 1, as above - so that a factor of p does exist, we can instead apply the difference of two squares to $1 - q^4$,to give:]

$4Kp(q^3 - 4q^3) + (4K)(1 - q^2)(1 + q^2)$

$= 4Kp(-3q^3) + 4K(1-q)(1+q)(1+q^2)$

$= 4Kp(-3q^3 + [1 + q^2 + q + q^3])$

$= 4Kp\{1 + q + q^2 - 2q^3\}$

E(cost for B) $= 6pq^5K + 15p^2q^4(2K)$

$+[1-6pq^5-15p^2q^4-q^6](6K)$

$= 6Kp(q^5+5pq^4-6q^5-15pq^4)+(1-q^6)(6K)$

[It will definitely be simpler to use the difference of two squares here, given that the answer has to be in terms of q.]

$= 6Kp(q^5+5(1-q)q^4-6q^5-15(1-q)q^4)$

$+(1-q^3)(1+q^3)(6K)$

$= 6Kp(q^5+5q^4-5q^5-6q^5-15q^4+15q^5)$

$+(1-q)(1+q+q^2)(1+q^3)(6K)$

[The STEP examiners are very fond of the factorisations

$a^3-b^3=(a-b)(a^2+ab+b^2)$ and $a^3+b^3=(a+b)(a^2-ab+b^2)$]

$= 6Kp\{5q^5-10q^4+(1+q+q^2)+(q^3+q^4+q^5)\}$

$= 6Kp\{1+q+q^2+q^3-9q^4+6q^5\}$

For the last part, we require

$4Kp\{1+q+q^2-2q^3\}=\frac{2}{3}.6Kp\{1+q+q^2+q^3-9q^4+6q^5\}$

$\Rightarrow -2q^3=q^3-9q^4+6q^5$

$\Rightarrow 3q^3(1-3q+2q^2)=0$

$\Rightarrow q=0$ or $(2q-1)(q-1)=0$

So $p=0,\frac{1}{2}$ or 1