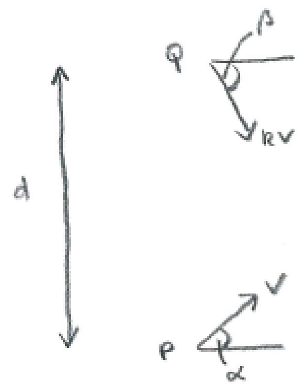
STEP 2009, Paper 1, Q9 - Solution (3 pages; 5/6/18)



[At first sight, it may seem that we cannot prove $\cos\alpha = \mathrm{k}\cos\beta$, since α, β & k can apparently be chosen at will. The problem is resolved by noting the other piece of information: that the particles collide. This places a constraint on α, β & k.]

Equating position vectors at time T:

$$\begin{pmatrix} V\cos\alpha\,.T \\ V\sin\alpha\,.T - \frac{1}{2}gT^2 \end{pmatrix} = \begin{pmatrix} kV\cos\beta\,.T \\ d - (kV\sin\beta\,.T + \frac{1}{2}gT^2) \end{pmatrix}$$

From the x component: $V\cos\alpha\,.T = kV\cos\beta\,.T$

so that $\cos\alpha = k\cos\beta$

From the y component: $V\sin\alpha\,.T - \frac{1}{2}gT^2 = d - (kV\sin\beta\,.T + \frac{1}{2}gT^2$

so that $V\sin\alpha\,.T = d - kV\sin\beta\,.T$ (A)

[Slightly worryingly, this is not a quadratic in T. It might seem that we have missed something, and this would probably be true at A Level. However, for STEP we can expect to have to do a fair amount of algebraic manipulation, so we shouldn't be too concerned. Although it would probably be more elegant to derive the required equation, it will be safer (and equally valid) to substitute our expression for T into the left-hand side of the equation, and show that this equals zero.]

From (A), $\mathrm{T}(Vsin\alpha + kVsin\beta) = \mathrm{d}$, and $\mathrm{T} = \frac{d}{Vsin\alpha + kVsin\beta}$ (B)

Then LHS of equation $= \frac{(k^2-1)V^2d^2}{(Vsin\alpha+kVsin\beta)^2} + \frac{2d^2Vsin\alpha}{Vsin\alpha+kVsin\beta} - d^2$

$= \frac{d^2(k^2-1+2sin\alpha(sin\alpha+ksin\beta)-(sin\alpha+ksin\beta)^2)}{(sin\alpha+ksin\beta)^2}$

As α and β are both between 0 and π, $sin\alpha + ksin\beta > 0$, so that we just need to show that

$k^2 - 1 + 2sin\alpha(sin\alpha + ksin\beta)$ - $(sin\alpha + ksin\beta)^2 = 0$ (C)

LHS of (B) $= k^2 - 1 + sin^2\alpha$ - $k^2sin^2\beta$ (D)

$cos\alpha = kcos\beta$ implies that 1 - $sin^2\alpha = k^2cos^2\beta$

Then (D) gives $k^2(1$ - $cos^2\beta + sin^2\beta) = 0$, as required.

The collision occurs when the y component of the velocity of P is zero;

ie when Vsinα – gT = 0 and $\mathrm{T} = \frac{Vsin\alpha}{g}$

Substituting for T in the equation in the question:

$\frac{(k^2-1)V^4sin^2\alpha}{g^2} + \frac{2dV^2sin^2\alpha}{g} - d^2 = 0$

and $sin^2\alpha\,\{(k^2-1)V^4 + 2\mathrm{d}V^2g\} = d^2g^2$

so that $sin^2\alpha = \frac{d^2g^2}{(k^2-1)V^4 + 2\mathrm{d}V^2g}$

[This is a useful reminder that some quite obscure expressions appear in STEP questions – ie you haven't necessarily made a mistake!]

The inequality in the final result can only come from the fact that $sin^2\alpha \geq 0$ or that $sin^2\alpha \leq 1$. The latter looks more promising and gives:

$d^2g^2 \leq (k^2 - 1)V^4 + 2dV^2g$ (E)

Let U $= \frac{gd}{V^2}$; we are then trying to show that U ≤ 1+k

(E) implies that U^2 – 2U + (1- k^2) ≤ 0 (F)

The equation U^2 – 2U + (1- k^2) = 0 has roots of $\frac{2\pm\sqrt{4-4(1-k^2)}}{2}$

$= 1\pm k$

(F) then implies that 1-k ≤ U ≤ 1+k

so that U ≤ 1+k, as required