STEP 2009, Paper 1, Q1 - Solution (2 pages; 5/6/18)

(i) The following are proper factors:

A: 3, 3x3

B: 5, 5x5, 5x5x5

plus: factors made up of 1 from each of A & B (eg 3x5x5); ie a further 2x3=6,

but excluding 3x3x5x5x5,

giving a total of $2 + 3 + 6 - 1 = 10$

In general, $3^m 5^n$ has $m + n + mn - 1$ proper factors

If $m + n + mn - 1 = 10$:

Suppose $m = 0$: then $n = 11$

Suppose $m = 1$: then $n = 5$

Suppose $m = 3$: then $n = 2$

Suppose $m = 4$: then $n = 7/5$ (but n has to be an integer)

Suppose $m = 5$: then $n = 1$

The next allowable value of n is 0, which occurs when m=11.

Thus there are 5 other integers satisfying the requirement.

(ii) [The number 426 may be rather off-putting, but its size actually helps to see what has to be done]

If we consider numbers with just 2 prime factors to start with, then we require $m + n + mn - 1 = 426$ (A)

The only way that we are going to be able to find m & n that satisfy this is if we can factorise the two sides.

Consider $(m+1)(n+1) = mn + m + n + 1$

This shows that (A) can be rewritten as (m+1)(n+1) – 2 = 426

and it now becomes clear that the number of proper factors can be written as (p+1)(q+1)(r+1)... – 2 in the case of more than 2 prime factors

[the expression (m+1)(n+1) - 2 could have been used in (i), and would have been more elegant]

Thus (p+1)(q+1)(r+1)... = 428 = 2x2x107 and 107 is prime, as it has no divisors ≤ 11 (we only need to consider divisors up to the square root of 107)

As 428 has just 3 prime factors, s, t ... equal 0, and

(p+1)(q+1)(r+1) = 2x2x107

The smallest number of the form $a^p b^q c^r$ will be either

2^{106}x3^3 (p=106,q=3, r=0) or 2^{106}x3x5 (p=106,q=1,r=1)

Of these, 2^{106}x3x5 is the smallest.