STEP 2009, Paper 1, Q12 - Solution (2 pages; 5/6/18)

[Inequalities and equations are often best rearranged into the form .... ≥ 0 etc.]

Consider $x^2 + y^2 - 2xy = (x-y)^2$ (A)

As (A) cannot be –ve, $x^2 + y^2 - 2xy \geq 0$ and hence $x^2 + y^2 \geq 2xy$

[With 'show / prove that' questions, the proof has to be completely convincing.]

(i)

	1	2
red	a	b
blue	b	a

Prob(same colour) $= \frac{a}{a+b} \text{ x } \frac{b}{a+b} + \frac{b}{a+b} \text{ x } \frac{a}{a+b} = \frac{2ab}{(a+b)^2}$

Prob(different colours) $= \frac{a}{a+b} \text{ x } \frac{a}{a+b} + \frac{b}{a+b} \text{ x } \frac{b}{a+b} = \frac{a^2+b^2}{(a+b)^2}$

Then , as $2ab \leq a^2+b^2$, Prob(same colour) $\leq$ Prob(different colours)

(ii)

	1	2	3
red	a	b	c
white	b	c	a
yellow	c	a	b

Prob(red/red/red') $= \frac{ab(a+b)}{(a+b+c)^3}$

So Prob(exactly 2 reds) $= 3.\frac{ab(a+b)}{(a+b+c)^3}$

and Prob(exactly 2 of the same colour)

$= \frac{3}{(a+b+c)^2}$ { ab(a+b) + ac(a+c) + bc(b+c) }

$= \frac{3}{(a+b+c)^2}$ $(a^2\text{b} + b^2\text{c} + c^2\text{a} + ab^2 + \text{b}c^2 + \text{c}a^2)$, as required

Prob(all red) $= \frac{abc}{(a+b+c)^3}$

So Prob(all the same colour) $= 3 \,.\, \frac{abc}{(a+b+c)^3}$

[Note that some parts of questions are not intended to be that difficult; they are just needed for a later part of the question.]

The last part follows if we can show that

$a^2\text{b} + b^2\text{c} + c^2\text{a} + ab^2 + \text{b}c^2 + \text{c}a^2 \geq 6\text{abc}$ (B)

[Clearly there is going to be some connection with the original inequality. (B) involves 3 letters and has cubic terms. We might investigate $(a + b + c)^2$ or $(a + b + c)^3$, but (apart from not leading anywhere), this is not really analogous to $(x - y)^2$, and we cannot introduce a minus sign (eg with $(a - b - c)^3$ without breaking the symmetry between the letters. This is one of those situations where we just have to look elsewhere, but at the same time keeping it simple.]

From the original inequality, $\text{c}(a^2 + b^2) \geq 2\text{abc}$ (C)

and so $\text{c}(a^2 + b^2) + \text{b}(a^2 + c^2) + \text{a}(b^2 + c^2) \geq 6\text{abc}$

which gives (B), so that

Prob(exactly 2 of the same colour) ≥ 6 Prob(all the same colour)

[(C) could perhaps be spotted by trying to create abc – as this is needed in (B)]