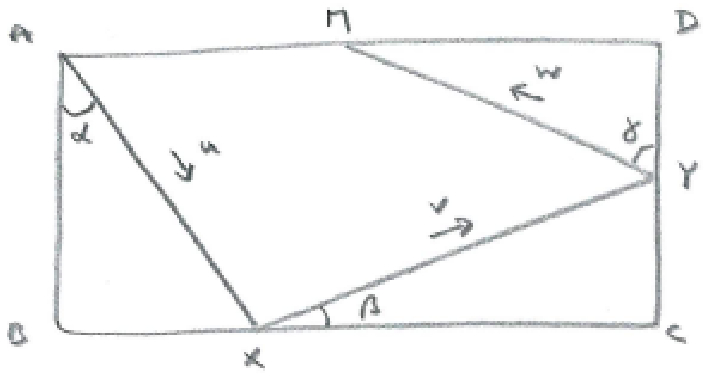
STEP 2008, Paper 2, Q10 – Solution (2 pages; 2/6/18)



At X, along the surface: $v\cos\beta = u\sin\alpha$ (CoLM) (A)

Perp. to the surface, $v\sin\beta = eu\cos\alpha$ (Newton's law of restitution) (B)

Hence, dividing (B) by (A), $\frac{e}{\tan\alpha} = \tan\beta$,

so that $\tan\alpha\tan\beta = e$, as required.

[The same process could be gone through for Y, but the examiners are unlikely to want to give marks for the same material, so it is worth looking for a shortcut.]

The same reasoning applies at Y, replacing α with β and β with γ, to give:

$\tan\beta\tan\gamma = e$, so that $\gamma = \alpha$ (given that both angles are less than 90^0).

[look out for refinements like this – in order to get full marks!]

(ii)

[The basic approach here is straightforward: create equations from the diagram and then solve them.]

$\tan\alpha = \frac{BX}{a}$ (C)

$\tan\beta = \frac{CY}{XC}$ (D)

$\tan\gamma$ (=$\tan\alpha$) $= \frac{b}{YD}$ (E)

Then (C)&(D) give: BX+XC = a $\tan\alpha + \frac{CY}{\tan\beta}$

so that 2b$\tan\beta$ = ae + CY (since $\tan\alpha\tan\beta$ = e) (F)

(E)&(F) then give: CY+YD = 2b$\tan\beta$ – ae + $\frac{b}{\tan\alpha}$

so that a$\tan\alpha$ = 2be – ae$\tan\alpha$ + b (again, since $\tan\alpha\tan\beta$ = e)

and a$\tan\alpha$(1+e) = b(2e+1)

and hence $\tan\alpha = \frac{(1+2e)b}{(1+e)a}$

The shot will be possible provided $0 < \tan\alpha < \frac{2b}{a}$ (G)

Now, $\frac{1+2e}{1+e} > 0$ (since e≥0) and $\frac{1+2e}{1+e} < \frac{2+2e}{1+e} = 2$

Thus (G) is satisfied and the shot is possible whatever the value of e.

(iii) Fraction of KE lost = 1 - $\frac{1/2.mw^2}{1/2.mu^2}$ (where m is the mass of the balls)

$= 1 - (\frac{w}{u})^2$

Now, v$\sin\beta$ = eu$\cos\alpha$ and w$\sin\alpha$ = ev$\cos\beta$ (since $\gamma = \alpha$)

so that w/u = $\frac{ev\cos\beta/\sin\alpha}{v\sin\beta/(e\cos\alpha)} = \frac{e^2 v\cos\beta\cos\alpha}{v\sin\beta\sin\alpha} = \frac{e^2}{\tan\beta\tan\alpha}$ = e and the fraction of KE lost = 1 - e^2

[This goes to show that the last part of a question – even a STEP 2 question – is not always any harder than the earlier parts.]