STEP 2007, Paper 3, Q7 – Solution (3 pages; 31/5/18)

In (ii), 'by making a substitution' in fact only involves the substitution that you have just been asked to look at (it seems too good to be true!)

Unusually, part (ii) doesn't seem to be needed for part (iii) – although part (i) is.

(i) Let $u = v^{-1}$, so that $du = -v^{-2}dv$

and $t(x) = -\int_{\infty}^{1/x} \frac{v^{-2}}{1+v^{-2}} dv = \int_{1/x}^{\infty} \frac{1}{v^2+1} dv$, which gives the required result.

Then $t\left(\frac{1}{x}\right) + t(x) = \int_{x}^{\infty} \frac{1}{1+v^2} dv + \int_0^x \frac{1}{1+u^2} du = \int_0^{\infty} \frac{1}{1+u^2} du = \frac{p}{2}$

and with $x = 1$, $2t(1) = \frac{p}{2}$

(ii) $y = \frac{u}{\sqrt{1+u^2}} \Rightarrow y^2 = \frac{u^2}{1+u^2} \Rightarrow \frac{1}{y^2} = \frac{1+u^2}{u^2} = \frac{1}{u^2} + 1$

$\Rightarrow \frac{1}{u} = \sqrt{\frac{1}{y^2} - 1} \Rightarrow u = \frac{1}{\sqrt{\frac{1-y^2}{y^2}}} = \frac{y}{\sqrt{1-y^2}}$

Then $\frac{du}{dy} = \frac{1}{1-y^2}\{\sqrt{1-y^2} - y\left(\frac{1}{2}\right)(1-y^2)^{-\frac{1}{2}}(-2y)\}$

$= \frac{1}{(1-y^2)^{\frac{3}{2}}}\{1 - y^2 + y^2\} = \frac{1}{\sqrt{(1-y^2)^3}}$, as required

[As we want to show that $t(x) = s\left(\frac{x}{\sqrt{1+x^2}}\right)$, it is a fairly safe bet that the upper limit of $\int_0^x \frac{1}{1+u^2} du$ needs to be transformed to $\frac{x}{\sqrt{1+x^2}}$; ie we want $u \to y = \frac{u}{\sqrt{1+u^2}}$ (so that when $u = x$,

$y = \frac{x}{\sqrt{1+x^2}}$)]

If $y = \frac{u}{\sqrt{1+u^2}}$, we have just shown that $u = \frac{y}{\sqrt{1-y^2}}$

and $du = \frac{1}{\sqrt{(1-y^2)^3}}dy$

Substituting into $\int_0^x \frac{1}{1+u^2}du$ then gives

$$\int_0^{\frac{x}{\sqrt{1+x^2}}} \frac{1}{(1+\frac{y^2}{1-y^2})} \frac{1}{\sqrt{(1-y^2)^3}}dy = \int_0^{\frac{x}{\sqrt{1+x^2}}} \frac{1}{(\frac{1}{1-y^2})} \frac{1}{\sqrt{(1-y^2)^3}}dy$$

$$\int_0^{\frac{x}{\sqrt{1+x^2}}} \frac{1}{\sqrt{(1-y^2)}}dy = s\left(\frac{x}{\sqrt{1+x^2}}\right)$$

And $x = 1 \Rightarrow t(1) = s\left(\frac{1}{\sqrt{2}}\right)$

and $2t(1) = \frac{p}{2}$ from (i), so that $s\left(\frac{1}{\sqrt{2}}\right) = \frac{p}{4}$, as required

(iii) Applying $t(x) = \int_0^x \frac{1}{1+u^2}du$,

$u = 0 \Rightarrow z = \frac{1}{\sqrt{3}}$

$u = \frac{1}{\sqrt{3}} \Rightarrow z = \frac{\left(\frac{2}{\sqrt{3}}\right)}{1-\frac{1}{3}} = \sqrt{3}$

and $\frac{dz}{du} = \frac{1}{\left(1-\frac{1}{\sqrt{3}}u\right)^2}\{\left(1-\frac{1}{\sqrt{3}}u\right) - \left(u+\frac{1}{\sqrt{3}}\right)\left(-\frac{1}{\sqrt{3}}\right)\}$

$= \frac{1+\frac{1}{3}}{\left(1-\frac{1}{\sqrt{3}}u\right)^2} = \frac{4}{3}\left(1-\frac{1}{\sqrt{3}}u\right)^{-2}$

Then $t\left(\frac{1}{\sqrt{3}}\right) = \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{3}{4}\left(1 - \frac{1}{\sqrt{3}}u\right)^2 . \frac{1}{1+u^2}\ dz$ (A)

Now, $1 + z^2 = \frac{\left(1-\frac{1}{\sqrt{3}}u\right)^2 + \left(u+\frac{1}{\sqrt{3}}\right)^2}{\left(1-\frac{1}{\sqrt{3}}u\right)^2} = \frac{\left(1-\frac{2}{\sqrt{3}}u+\frac{1}{3}u^2\right)+(u^2+\frac{1}{3}+\frac{2}{\sqrt{3}}u)}{\left(1-\frac{1}{\sqrt{3}}u\right)^2}$

$= \frac{\frac{4}{3}(u^2+1)}{\left(1-\frac{1}{\sqrt{3}}u\right)^2}$, so that $\frac{1}{1+z^2} = \frac{3}{4}\left(1 - \frac{1}{\sqrt{3}}u\right)^2 (\frac{1}{1+u^2})$

and hence (A)$\Rightarrow t\left(\frac{1}{\sqrt{3}}\right) = \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{1+z^2}\ dz$, as required.   (B)

Also $\frac{p}{2} = \int_0^\infty \frac{1}{1+u^2} du = \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{1+u^2} du + \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{1+u^2} du + \int_{\sqrt{3}}^{\infty} \frac{1}{1+u^2} du$

$= t\left(\frac{1}{\sqrt{3}}\right) + t\left(\frac{1}{\sqrt{3}}\right) + t\left(\frac{1}{\sqrt{3}}\right)$,

from the definition of $t(x)$, (B) and (i), respectively.

Thus $3t\left(\frac{1}{\sqrt{3}}\right) = \frac{p}{2}$, as required.