STEP 2007, Paper 2, Q1 – Solution (2 pages; 23/5/18)

(i) $(1+\frac{k}{100})^{1/2} = 1 + \frac{1}{2}\left(\frac{k}{100}\right) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{k}{100}\right)^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}\left(\frac{k}{100}\right)^3$

$= 1 + \frac{k}{200} - \frac{k^2}{80000} + \frac{k^3}{16000000} + \cdots$

(a) $(1+\frac{8}{100})^{1/2} = \frac{108^{1/2}}{10} = \frac{\sqrt{12\times 9}}{10} = \frac{3(2)\sqrt{3}}{10} = \frac{3\sqrt{3}}{5}$

so that , from (i), $\sqrt{3} \approx \frac{5}{3}\{1 + \frac{8}{200} - \frac{64}{80000} + \frac{8^3}{16000000}\}$

$= \frac{5}{3}\{1 + 0.04 - 0.0008 + \frac{32}{1000000}\}$

$= \frac{1}{3}\{5 + 0.2 - 0.004 + 0.00016\}$

$= \frac{1}{3}(5.20016 - 0.004)$

$= \frac{1}{3}(5.19616)$

$= 1.73205$ (5dp)

(b) We want an integer k whose absolute value is as small as possible, such that $100 + k = 6n^2$, for some integer n

eg $k = -4$, with $n = 4$

Then $(1+\frac{-4}{100})^{1/2} = \frac{4\sqrt{6}}{10}$

so that $\sqrt{6} \approx \frac{10}{4}\{1 - \frac{4}{200} - \frac{16}{80000} - \frac{64}{16000000}\}$

$= 2.5 - 0.05 - 0.0005 - 0.00001$

$= 2.5 - 0.05051 = 2.44949$

(ii) $(1+\frac{k}{1000})^{1/3} \approx 1 + \frac{1}{3}\left(\frac{k}{1000}\right)$

We now want an integer k whose absolute value is as small as possible, such that $1000 + k = 3n^3$, for some integer n

In order to find a small k, consider $1000 = 3x^3$

$x = 6 \Rightarrow 3x^3 = 648$, whilst $x = 7 \Rightarrow 3x^3 = 1029$

So let $k = 29$, to give:

$(\frac{1029}{1000})^{1/3} \approx 1 + \frac{1}{3}\left(\frac{29}{1000}\right)$

and $\frac{7\sqrt[3]{3}}{10} \approx \frac{3029}{3000}$,

so that $\sqrt[3]{3} \approx \frac{3029}{2100}$, as required.