STEP 2007, Paper 1, Q5 – Solution (5 pages; 21/5/18)

Possible approaches to finding angles between planes:

(A) (long) vector method: find the vector equations of the planes, and convert to Cartesian form, in order to find the direction vectors (perpendicular to the planes), and then obtain $cos\theta$ via the scalar product.

(B) Look for a triangle, where one side is a line in one plane, and another side is a line in the other plane, such that the normals to the two planes at the two lines are in the plane of the triangle (see triangle T in Figure 2).

[We can't just find the angle between any line in one plane and any line in the other: for example, the join between two adjacent faces of the octahedron are lines in both planes, and the angle between the lines in this case is 0°!]

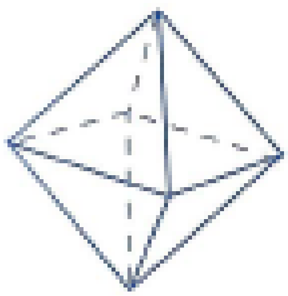

Figure 1

For (i), we can treat the octahedron as two square-based pyramids stuck together at their bases, with one pyramid pointing directly upwards (almost as in Figure 1).

All pairs of (adjacent) faces will have the same angle between them (as it's a regular octahedron), but the pairs for which the angle can most easily be determined are where one face is in the upper pyramid and the other is in the lower pyramid.

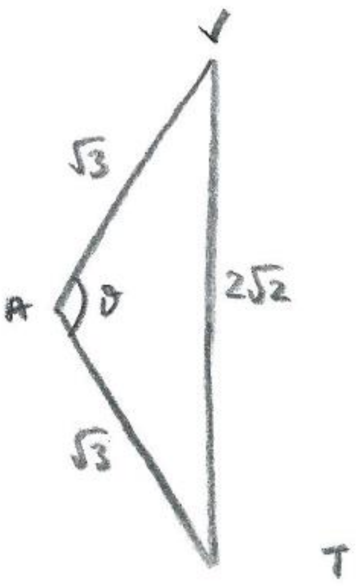

Figure 2

This enables an isosceles triangle (T) to be created (see Figure 2), with one side being the the line joining the top and bottom vertices, and the other (equal) sides being the medians of the faces in question (drawn from the vertices). [A median is a line from one vertex of a triangle to the mid-point of the opposite side.] Thus T is obtained from a vertical cross-section of the octahedron.

Without loss of generality, we can assume that the sides of the equilateral triangles forming the faces have length 2. The medians leading to the vertex then have length $\sqrt{3}$.

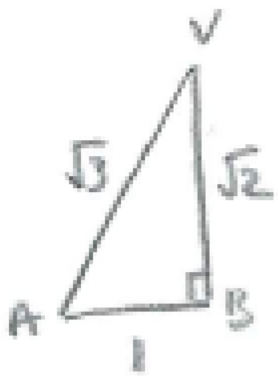

Figure 3

Referring to figure 3, we can form the right-angled triangle with corners at the vertex (V), the centre of the base of the top pyramid (B), and the base of a median (A). By Pythagoras, $VB = \sqrt{2}$.

The triangle T, referred to above, thus has sides $2\sqrt{2}, \sqrt{3}$ & $\sqrt{3}$, and the angle between the two equal sides, which is the required angle, can be found from the Cosine rule (as in the official solution, though with $k = 1$).

Thus: $\left(2\sqrt{2}\right)^2 = \left(\sqrt{3}\right)^2 + \left(\sqrt{3}\right)^2 - 2\left(\sqrt{3}\right)\left(\sqrt{3}\right)cos\theta$

so that $8 = 6 - 6cos\theta$, and hence $cos\theta = -\frac{2}{6} = -\frac{1}{3}$

Alternatively, vectors can be used, as follows:

Create x and y axes along the bottom of two faces of the upper pyramid, with z being vertical (so that the origin is at one corner of the base of the pyramid).

Then a vector equation of the plane containing one face and the y-axis is:

$$\underline{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ \sqrt{2} \end{pmatrix},$$

since the plane contains the origin, the point (0,2,0) and the point $(1,1,\sqrt{2})$, which is V (ie the 3 corners of the face).

Converting to a Cartesian equation, we have

$x = \mu$, $y = 2\lambda + \mu$ and $z = \mu\sqrt{2}$

so that $z = x\sqrt{2}$, and the equation can be written as $\sqrt{2}x + 0y - z = 0$.

Hence this face of the pyramid has direction vector $\begin{pmatrix} \sqrt{2} \\ 0 \\ -1 \end{pmatrix}$

or $\begin{pmatrix} -\sqrt{2} \\ 0 \\ 1 \end{pmatrix}$, to ensure that it is pointing away from the inside of the pyramid.

Similarly, the direction vector of the plane containing one face and the x-axis is $\begin{pmatrix} 0 \\ -\sqrt{2} \\ 1 \end{pmatrix}$.

The angle between the outward-pointing direction vectors of these faces is then given by

$$cos\theta = \frac{\begin{pmatrix} -\sqrt{2} \\ 0 \\ 1 \end{pmatrix} . \begin{pmatrix} 0 \\ -\sqrt{2} \\ 1 \end{pmatrix}}{\left|\begin{pmatrix} -\sqrt{2} \\ 0 \\ 1 \end{pmatrix}\right|\left|\begin{pmatrix} 0 \\ -\sqrt{2} \\ 1 \end{pmatrix}\right|} = \frac{1}{3}$$

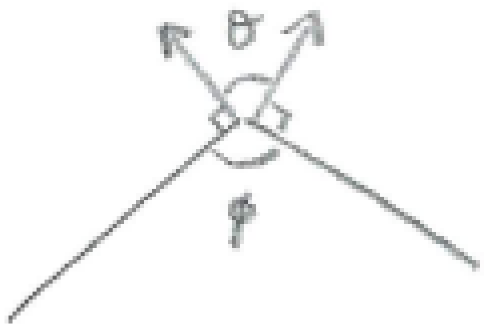

Figure 4

Referring to Figure 4, the angle that we require is $\phi = 180 - \theta$, and $cos\phi = -cos\theta = -\frac{1}{3}$

For (ii), the official solutions mention a proof of the fact that the centre of an equilateral triangle occurs 2/3 of the way along each median (from the vertex). The ratio of $1: cos60°$ that is referred to can be found from Figure 5, where $x = 1cos60°$.

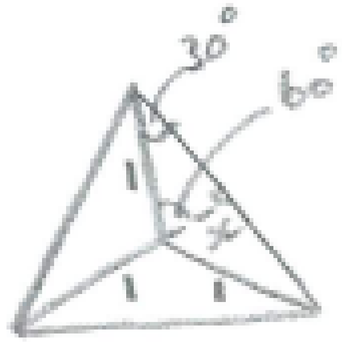

Figure 5

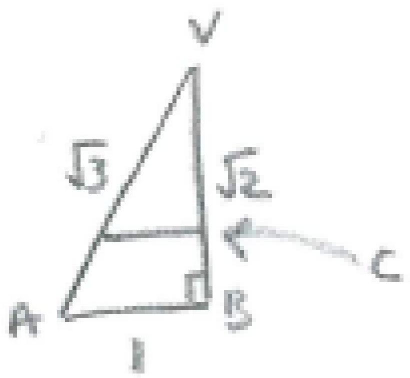

Figure 6

Referring to Figure 6, a slightly different way of finding the side of the cube is to note that the height of the cube will be twice BC, and that, by similar triangles, $BC = \frac{1}{3}\sqrt{2}$ (since the centre of the face is 1/3 of the way along AV).

The volume of the octahedron is twice the volume of the square-based pyramid (1/3 × base × height).