STEP 1, 2007 – Notes (6 pages; 23/5/18)

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Q6 This question has a high proportion of 'show that' results, or results that can be checked.

The expansion of x^3 - y^3 (as (x-y)(x^2+xy+y^2)) is a favourite of the examiners, so it's a fair bet that the question has been created by using this fact (though you don't in fact need to use it).

It is possible to use a symmetry argument, once x has been expressed in terms of d: we can re-write x^2 - $y^2 = (x-y)^3$ as y^2 - $x^2 = (y-x)^3$ with y – x = D (where D = -d). Then x = $\frac{d(d+1)}{2}$ becomes y = $\frac{D(D+1)}{2}$ etc.

Q7 (i) L_1 is $\underline{r} = \begin{pmatrix} 1+2\lambda \\ 2\lambda \\ 2-3\lambda \end{pmatrix}$ & L_2 is $\underline{r} = \begin{pmatrix} 4+\mu \\ 2\mu-2 \\ 9-2\mu \end{pmatrix}$

$D^2 = (1+2\lambda-[4+\mu])^2 + (2\lambda-[2\mu-2])^2 + (2-3\lambda-[9-2\mu])^2$

$= (-3+2\lambda-\mu)^2 + 4(\lambda-\mu+1)^2 + (-7-3\lambda+2\mu)^2$

$= (9+4+49) + \lambda^2(4+4+9) + \mu^2(1+4+4)$

$+2\lambda(-6+4+21)+2\mu(3-4-14)+2\lambda\mu(-2-4-6)$

$=62+17\lambda^2+9\mu^2+38\lambda-30\mu-24\lambda\mu$

$=(\lambda-1)^2+36+[25+40\lambda+16\lambda^2+9\mu^2-30\mu-24\lambda\mu\,]$

and then $(3\mu-4\lambda-5)^2=9\mu^2+16\lambda^2+25-24\lambda\mu-30\mu+40\lambda$
,

which equals the expression in the square brackets, so that the required result follows

ie $D^2=(3\mu-4\lambda-5)^2+(\lambda-1)^2+36$

D^2 is minimised when the two squared terms are both zero, giving

$D=\sqrt{36}=6$

At this point, $\lambda=1$ & $3\mu-4(1)-5=0$, so that $\mu=3$.

Then the two coordinates are $\begin{pmatrix}1+2(1)\\2(1)\\2-3(1)\end{pmatrix}=\begin{pmatrix}3\\2\\-1\end{pmatrix}$

& $\begin{pmatrix}4+3\\2(3)-2\\9-2(3)\end{pmatrix}=\begin{pmatrix}7\\4\\3\end{pmatrix}$

[strictly speaking, these should be written as $(3,2,-1)$& $(7,4,3)$]

(ii) As before, we obtain

$D^2=(2-3-4\beta k)^2+(3+\alpha-3-\beta+\beta k)^2+(5+2+3\beta k)^2$

$=(1+4\beta k)^2+(\alpha-\beta+\beta k)^2+(7+3\beta k)^2$

[The question now is whether we need to expand this and look for a suitable arrangement into two squares plus a constant number, or is there a shortcut? The former method has several drawbacks:

(a) it is time-consuming, (b) we might not find a suitable arrangement, and (c) we don't know whether more than one rearrangement is possible: if there is only one, it seems quite possible that we might not discover it! For these reasons, we can probably reject the latter approach as too risky, and concentrate on the former. The official solution uses the risky approach, but without explaining where the inspiration comes from when deriving the squared terms, beyond saying - rather unconvincingly - (in the examiners' report) that "the coefficients do not permit many possibilities". The approach adopted below is much simpler.]

The reason why, in (i), $D^2 = (3\mu - 4\lambda - 5)^2 + (\lambda - 1)^2 + 36$ is an advantageous form is that only one squared term involves both λ & μ.

Note that in (ii) only the middle squared term involves both α & β (whereas each of the squared terms at the corresponding stage in (i) involve both λ & μ). So there is no need to expand everything: we can just expand the 1st and 3rd squared terms, and complete the square.

Thus $(1 + 4\beta k)^2 + (7 + 3\beta k)^2 = \beta^2(16k^2 + 9k^2) + \beta(8k + 42k) + 50$

$= 25k^2\beta^2 + 50k\beta + 50 = (5k\beta + 5)^2 + 25$

and $D^2 = (\alpha - \beta + \beta k)^2 + (5k\beta + 5)^2 + 25$,

giving a minimum distance of $\sqrt{25} = 5$, provided that $k \neq 0$. When $k = 0$, the minimum distance is $\sqrt{50}$.

When $k = 0$, both L_3 & L_4 have a direction vector of $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$; ie they are both parallel to the y-axis. When $k \neq 0$, the two lines are skew (ie not parallel and not intersecting).

Q8 The points where the two curves touch and intersect can be dealt with together, by equating the cubics and factorising with a factor of $(x-2)^2 = x^2 - 4x + 4$ (so that the other factor will have to be (a-1)x – (2a+4) ; we just need to expand out to check).

Q9 As the situation in the question is fairly straightforward M1 material, this question mainly concerns algebraic manipulation.

A useful 'sledgehammer' method (straightforward, but not always the most elegant) for eliminating a variable from two equations (in this case, W) is just to make W the subject of each equation, and equate the resulting expressions.

Q10 Two devices that can be useful in this sort of situation are:

(i) to consider extremes; eg large y, small x ; this can enable the possible scenarios to be established

(ii) to consider 'critical situations': where different scenarios meet; this can explain the significance of the inequalities in the question

The equations of lines are probably most simply derived in the form

y = mx +c (rather than from 2 points on the line). In order to obtain c for lines representing the return journeys of the horsemen, the lines representing the outward journeys can be reflected in the x-axis.

Q11 Conservation of Energy can be used as an alternative to N2L for the first part, to give $u^2 = {u_1}^2 + gL$, where u_1 is the speed at the end of the tube. The algebra can be shortened (a bit) by only substituting for u_1 at the end.

In order to show that $\frac{dR}{dL} = 0$ when $2D = L\sqrt{3}$, there doesn't seem to be any reason why $L\sqrt{3}$ can't just be substituted for 2D in the expression for $\frac{dD}{dL}$.

The Examiners' Report says that hardly any attempts were made at this question. This was presumably because of the complicated looking expressions that had to be derived. However, note that (a) everything but the last part is of the "show that" form, and (b) the topic of projectiles usually involves nothing more complicated than a quadratic equation.

Q13 This question illustrates that, for probability questions, there is often more than one possible approach. Having found a workable method, it's worth looking for another one - as it may be either quicker or less prone to errors.

In general, common approaches are:

(a) case by case (using conditional probabilities); with or without a tree diagram

(b) $\frac{\text{no. of favourable outcomes}}{\text{no. of possible outcomes}}$ (provided the outcomes are equally likely)

(combinations can often be used, as in this question)

(c) Venn diagram equations:

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cap B) = P(A) \times P(B|A)$

Q14 The Examiners' Report says that only a few attempts at this question were seen (for the whole country presumably). However, it only really requires knowledge of the definition of the Poisson distribution (this appears in S2, but might be worth covering for the STEP exam).

For the last part, an alternative (and arguably simpler) solution is just to treat the resulting equation linking λ and r as a quadratic equation in λ, and then apply the usual $b^2 - 4ac = 0$.