STEP 2006, Paper 3, Q3 - Solution (3 pages; 19/5/18)

(i) $tanx$ is an odd function; ie $tan(-x) = -tanx$ for all x (1)

If $a_n \neq 0$ for some even n, then $a_n(-x)^n = a_n x^n$, and contradicting (1), except possibly for isolated values of x.

[A rigorous proof is unlikely to be needed here: the H&A just state that $tanx$ is an odd function.]

The identity is equivalent to $cotx - tanx - 2cot2x \equiv 0$

LHS $= \frac{1}{tanx} - tanx - \frac{2(1-tan^2x)}{2tanx}$

$= \frac{1}{tanx}\{1 - tan^2x - 1 + tan^2x) = 0$ unless $tanx = 0$

However, the original identity to be proved is undefined if $tanx = 0$, so this possibility can be excluded.

$cotx - tanx = 2cot2x$

$\Rightarrow \frac{1}{x} + \sum_{n=0}^{\infty} b_n x^n - \sum_{n=0}^{\infty} a_n x^n = \frac{2}{2x} + 2\sum_{n=0}^{\infty} b_n (2x)^n$

Equating coefficients of x^n,

$b_n - a_n = 2^{n+1} b_n$,

so that $a_n = (1 - 2^{n+1})b_n$, as required.

(ii) $cotx + tanx = \frac{cosx}{sinx} + \frac{sinx}{cosx} = \frac{cos^2x + sin^2x}{sinxcosx} = \frac{2}{\sin(2x)}$

$= 2cosec(2x)$

Hence $\frac{1}{x} + \sum_{n=0}^{\infty} b_n x^n + \sum_{n=0}^{\infty} a_n x^n = \frac{2}{2x} + 2\sum_{n=0}^{\infty} c_n (2x)^n$

Equating coefficients of x^n,

$b_n + a_n = 2^{n+1}c_n$,

so that $c_n = 2^{-n-1}(b_n + [1 - 2^{n+1}]b_n)$, from (i).

Thus $c_n = 2^{-n-1}b_n(2 - 2^{n+1}) = b_n(2^{-n} - 1)$, as required.

(iii) $(1 + x\sum_{n=0}^{\infty} b_n x^n)^2 + x^2 - (1 + x\sum_{n=0}^{\infty} c_n x^n)^2$

$= (1 + xcotx - 1)^2 + x^2 - (1 + xcosecx - 1)^2$

$$= \frac{x^2cos^2x}{sin^2x} + x^2 - \frac{x^2}{sin^2x} = \frac{x^2}{sin^2x}\{cos^2x + sin^2x - 1\} = 0,$$

giving the required result.

[Expanding both sides of the equation in (iii) and equating coefficients is one option, but use of the difference of two squares looks promising.]

$(1 + x\sum_{n=0}^{\infty} c_n x^n)^2 - (1 + x\sum_{n=0}^{\infty} b_n x^n)^2 = x^2$

$\Rightarrow (2 + x\sum_{n=0}^{\infty}(c_n + b_n)x^n)(x\sum_{n=0}^{\infty}(c_n - b_n)x^n) = x^2$

$\Rightarrow (2 + x\sum_{n=0}^{\infty} b_n 2^{-n}x^n)(x\sum_{n=0}^{\infty}(2^{-n} - 2)b_n x^n) = x^2$

$\Rightarrow (2 + x\sum_{n=0}^{\infty} b_n 2^{-n}x^n)(\sum_{n=0}^{\infty}(2^{-n} - 2)b_n x^n) = x$

As $a_n = 0$ for even n, $a_n = (1 - 2^{n+1})b_n \Rightarrow b_n = 0$ for even n also, as $1 - 2^{n+1} \neq 0$

So $\left(2 + \frac{1}{2}b_1x^2 + \frac{1}{8}b_3x^4 + \cdots\right)\left(-\frac{3}{2}b_1x - \frac{15}{8}b_3x^3 + \cdots\right) = x$

and $\left(2 + \frac{1}{2}b_1x^2 + \frac{1}{8}b_3x^4 + \cdots\right)\left(-\frac{3}{2}b_1 - \frac{15}{8}b_3x^2 + \cdots\right) = 1$

Then equating the constant terms gives $2\left(-\frac{3}{2}b_1\right) = 1$, so that $b_1 = -\frac{1}{3}$ and $a_1 = (1-4)\left(-\frac{1}{3}\right) = 1$, as required, from (i).

Equating the coeff. of x^2 gives: $2\left(-\frac{15}{8}\right)b_3 + \frac{1}{2}b_1\left(-\frac{3}{2}b_1\right) = 0$,

so that $5b_3 + {b_1}^2 = 0$,

and hence $b_3 = -\frac{1}{5}\left(-\frac{1}{3}\right)^2 = -\frac{1}{45}$

giving $a_3 = (1-16)\left(-\frac{1}{45}\right) = \frac{1}{3}$