STEP 2006, Paper 3, Q13 – Solution (3 pages; 20/5/18)

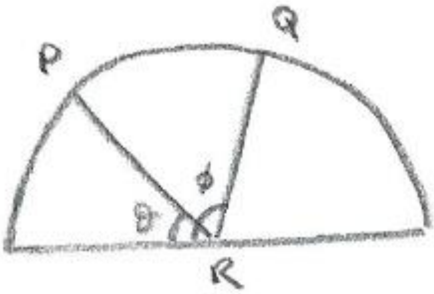

Fig. 1

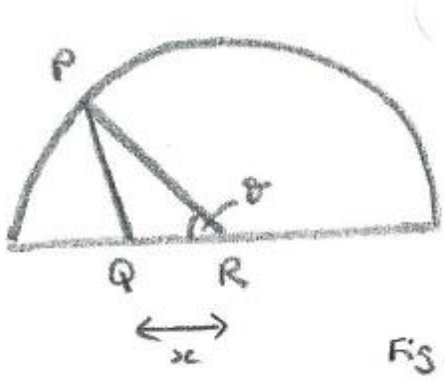

Fig. 2

Case 1: P & Q both lie on the arc

Case 2: one of them lies on the arc & the other lies on the diameter

(The case where they both lie on the diameter gives $A = 0$)

Given that Case 1 applies, the joint pdf of θ & ϕ (the angles made by P & Q respectively, as in Fig. 1, is that of a 2D uniform distribution with each variable ranging from 0 to π, so that $f(\theta, \phi) = \frac{1}{\pi^2}$

As $\theta < \phi$ half of the time, and this scenario contributes half of E(A) for Case 1, we can carry out the integration on the basis that $\theta < \phi$, and then double the result.

$A = \frac{1}{2}(1)(1)sin(\phi - \theta)$, so that the contribution to E(A) for Case 1 is

$$2\int_{\phi=0}^{\pi}\int_{\theta=0}^{\phi}\frac{1}{\pi^2}\left(\frac{1}{2}\right)sin(\phi-\theta)d\theta d\phi = \frac{1}{\pi^2}\int_{\phi=0}^{\pi}[cos(\phi-\theta)]_{\theta=0}^{\phi}d\phi$$

$$= \frac{1}{\pi^2}\int_{\phi=0}^{\pi}1 - cos\phi\ d\phi = \frac{1}{\pi^2}[\phi - sin\phi]_0^{\pi} = \frac{1}{\pi^2}(\pi) = \frac{1}{\pi}$$

[See below for alternative approach.]

Given that Case 2 applies, consider separately the 4 sub-cases:

(i) P on the arc, Q to the left of R

(ii) P on the arc, Q to the right of R

(iii) Q on the arc, P to the left of R

(iv) Q on the arc, P to the right of R

Given that (i) applies (see Fig. 2), $A = \frac{1}{2}(1)(x)sin\theta$

and the joint pdf will be $\frac{1}{\pi}$ (1) , as x is uniform over 0 to 1

The contribution to E(A) from (i) is then

$$\int_{\theta=0}^{\pi}\int_{x=0}^{1}\frac{1}{\pi}\left(\frac{1}{2}xsin\theta\right)dxd\theta = \frac{1}{2\pi}\int_{\theta=0}^{\pi}\left[\frac{1}{2}x^2\right]_0^1 sin\theta d\theta$$

$$= \frac{1}{4\pi}\int_0^{\pi} sin\theta d\theta = \frac{1}{4\pi}[-cos\theta]_0^{\pi} = \frac{1}{4\pi}\big(1-(-1)\big) = \frac{1}{2\pi}$$

For (ii), $A = \frac{1}{2}(1)(x)sin(\pi-\theta) = \frac{1}{2}(1)(x)sin(\theta)$, giving the same contribution to E(A).

Similarly for (iii) & (iv), so that the total contribution to E(A) for Case 2 is $\frac{4}{2\pi} = \frac{2}{\pi}$

Now P(Case 1) $= \left(\frac{\pi}{\pi+2}\right)^2$ (considering the lengths of the arc and the diameter) and

P(each of the 4 sub-cases for Case 2) $= \left(\frac{\pi}{\pi+2}\right)\left(\frac{1}{\pi+2}\right) = \frac{\pi}{(\pi+2)^2}$,

so that the overall E(A) $= \left(\frac{\pi}{\pi+2}\right)^2\left(\frac{1}{\pi}\right) + \frac{\pi}{(\pi+2)^2}\left(\frac{2}{\pi}\right)$

$= \frac{\pi+2}{(\pi+2)^2} = \frac{1}{\pi+2} = (2+\pi)^{-1}$, as required.

Alternative approach for Case 1: Following the method in the H&A, we can suppose for the moment that P lies in the interval $(\theta, \theta + \delta\theta)$, whilst Q lies in $(\phi, \phi + \delta\phi)$. Given this situation,

$A = \frac{1}{2}\sin(\theta - \phi)$ if $\phi < \theta$ and $A = \frac{1}{2}\sin(\phi - \theta)$ if $\phi > \theta$

[In the H&A, for case (i) the area is stated to be $\frac{1}{2}$|r|sinθ, but this seems to be mis-copied from case (ii).]

Given that P lies in $(\theta, \theta + \delta\theta)$,

$E(A) = \int_0^\theta \frac{1}{2}\sin(\theta - \phi)\left(\frac{1}{\pi}\right) d\phi + \int_\theta^\pi \frac{1}{2}\sin(\phi - \theta)\left(\frac{1}{\pi}\right) d\phi$

(since Q is uniformly distributed over $(0, \pi)$)

$= \frac{1}{2\pi}[\cos(\theta - \phi)]_{\phi = 0}^{\theta} + \frac{1}{2\pi}[-\cos(\phi - \theta)]_{\phi = \theta}^{\pi}$

$= \frac{1}{2\pi}(1 - cos\theta) + \frac{1}{2\pi}(-\cos(\pi - \theta) + 1) = \frac{1}{\pi}$,

as $\cos(\pi - \theta) = -cos\theta$

As $E(A)$ is independent of θ, its value is therefore $\frac{1}{\pi}$, given that Case 1 applies.