STEP 2006, P2, Q2 – Solution (3 pages; 16/5/18)

$e = e^1 = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \cdots > 2\frac{2}{3} = \frac{8}{3}$, as required

To show that $n! > 2^n$ for $n \geq 4$:

Proof by induction

When $n = 4$, $n! = 24 > 16 = 2^n$. Thus the result is true for $n = 4$.

Assume that the result is true for $n = k$, so that $k! > 2^k$

Then $\frac{(k+1)!}{2^{k+1}} = \frac{(k+1)k!}{2(2^k)} > \frac{k+1}{2}$, since $k! > 2^k$

As $\frac{k+1}{2} > 1$ when $k \geq 4$, it follows that $\frac{(k+1)!}{2^{k+1}} > 1$

and hence $(k+1)! > 2^{k+1}$

So if the result is true for $n = k$, it is true for $n = k + 1$.

As the result is true for $n = 4$, it is therefore true for $n = 5,6, \ldots$,
and hence, by the principle of induction, for all integer $n \geq 4$.

Then $e < 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{2^4} + \frac{1}{2^5} + \cdots$
$\frac{8}{3} + \frac{1}{16} \cdot \frac{1}{1-1/2}$ (sum of an infinite geometric series)

$= \frac{8}{3} + \frac{1}{8} = \frac{67}{24}$

$y = 3e^{2x} + 14ln\left(\frac{4}{3} - x\right)$

$\frac{dy}{dx} = 6e^{2x} - \frac{14}{\frac{4}{3}-x}$

At a stationary point, $\frac{dy}{dx} = 0$

so that $6e^{2x} - \frac{42}{4-3x} = 0$ (1)

$x = \frac{1}{2} \Rightarrow LHS\ of\ (1) = 6e - \frac{42}{4-\frac{3}{2}} = 6\left(e - \frac{7}{\left(\frac{5}{2}\right)}\right) = 6\left(e - \frac{14}{5}\right)$

[note at this point that $e = 2.718 \ldots < 2.8$]

$< 6\left(\frac{67}{24} - \frac{14}{5}\right) = 6\frac{(335-240-96)}{120} = \frac{(335-336)}{20} < 0$

$x = 1 \Rightarrow LHS\ of\ (1) = 6e^2 - 42 > 6\left(\frac{64}{9} - 7\right) = \frac{6}{9}(64 - 63) > 0$

Thus there is a root of (1) between 1/2 and 1 (since $\frac{dy}{dx}$ is continuous in this interval), and hence a stationary point occurs between 1/2 and 1.

Also, as $\frac{dy}{dx} < 0$ to the left of the stationary point, and $\frac{dy}{dx} > 0$ to the right of the stationary point, we can conclude that the stationary point is a minimum turning point.

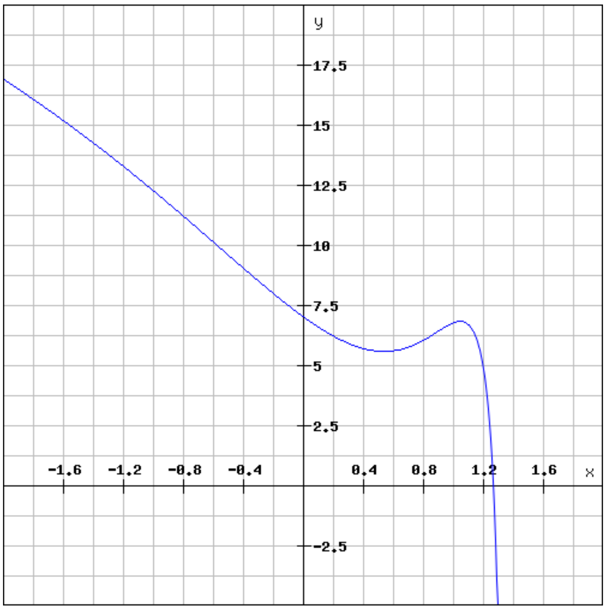
[Strictly speaking, there could be more than one stationary point between 1/2 and 1, but one of them must be a minimum in order for $\frac{dy}{dx}$ to change from being -ve to being +ve]

$x = 0 \Rightarrow y = 3 + 14ln(4/3) > 3$

As $x \to -\infty, y \to 0 + \infty$

As $x \to \frac{4}{3}, y \to 3e^{8/3} - \infty$

Hence there must be a maximum turning point between 1 and 4/3.



y
17.5
15
12.5
10
7.5
5
2.5
-2.5
-1.6
-1.2
-0.8
-0.4
0.4
0.8
1.2
1.6
x