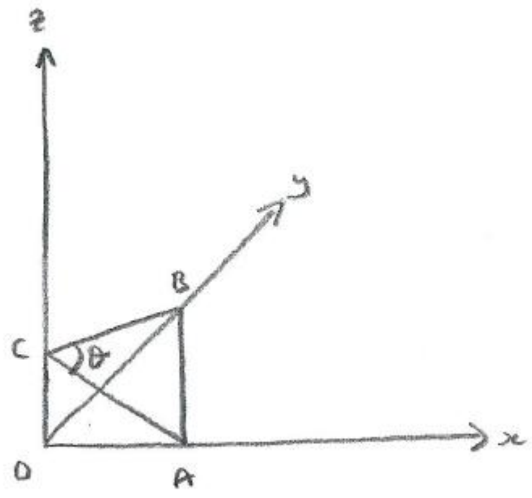
STEP 2006, Paper 1, Q8 - Solution (2 pages; 14/5/18)



(i) Taking OAB as the base, volume $= \frac{1}{3}\left(\frac{1}{2}ab\right)c = \frac{1}{6}abc$

(ii) By the Cosine rule, $AB^2 = AC^2 + BC^2 - 2AC.AB.cos\theta$

$\Rightarrow cos\theta = \frac{AC^2+BC^2-AB^2}{2AC.AB} = \frac{(a^2+c^2)+(b^2+c^2)-(a^2+b^2)}{2\sqrt{(a^2+c^2)(b^2+c^2)}} = \frac{c^2}{\sqrt{(a^2+c^2)(b^2+c^2)}}$

as required. (Using the scalar product is a bit quicker, as in the H&A.)

Area of ABC $= \frac{1}{2}AC.BCsin\theta$

$sin^2\theta = 1 - \frac{c^4}{(a^2+c^2)(b^2+c^2)} = \frac{a^2b^2+a^2c^2+c^2b^2}{(a^2+c^2)(b^2+c^2)}$

so that Area of ABC $= \frac{1}{2}\sqrt{(a^2+c^2)(b^2+c^2)}\,.\frac{\sqrt{a^2b^2+a^2c^2+c^2b^2}}{\sqrt{(a^2+c^2)(b^2+c^2)}}$

$= \frac{1}{2}\sqrt{a^2b^2+a^2c^2+c^2b^2}$

Taking ABC as the base, volume $= \frac{1}{3}\left(\frac{1}{2}\sqrt{a^2b^2+a^2c^2+c^2b^2}\right)d$

Then, from (i), $\frac{1}{3}\left(\frac{1}{2}\sqrt{a^2b^2+a^2c^2+c^2b^2}\right)d = \frac{1}{6}abc$, so that $(a^2b^2 + a^2c^2 + c^2b^2)d^2 = a^2b^2c^2$

and $\frac{1}{d^2} = \frac{(a^2b^2+a^2c^2+c^2b^2)}{a^2b^2c^2} = \frac{1}{c^2} + \frac{1}{b^2} + \frac{1}{a^2}$,

giving the required answer.