STEP 2006, Paper 1, Q4 – Solution (3 pages; 13/5/18)

[There are a couple of 'typos' in the H&As - see below.]

[Radians must be used here, in order for the gradient of $y = sinx$ to tend to 1 as $x \to 0$]

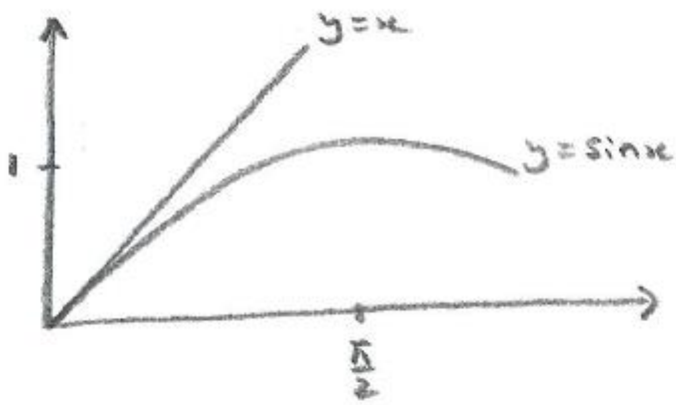

Fig. 1

(i) The gradient of $y = sinx$ at $x = 0$ is $cos(0) = 1$, and for

small $x > 0$ it is $cosx < 1$ and ≤ 1 for larger values of x; ie the graph of $y = sinx$ falls below that of $y = x$ and is never steeper than it, so that $x > sinx$ for $x > 0$

(ii) For small x, the graphs of $y = sinx$ and $y = x$ have approximately the same gradient of 1 (as $cosx \approx 1$ for small x), and, as they both pass through the Origin, it follows that $sinx \approx x$, and hence $\frac{sinx}{x} \approx 1$ for small x

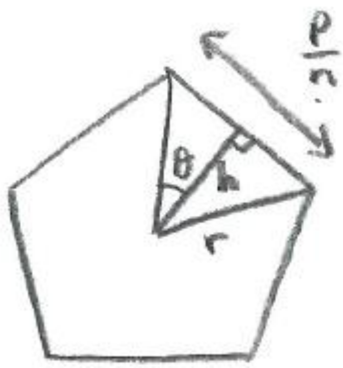

Fig. 2

The regular pentagon shown in Fig. 2 is representative of a general polygon. The area of the triangle shown is $\frac{1}{2}\left(\frac{P}{n}\right)h$

where $htan\theta = \frac{1}{2}\left(\frac{P}{n}\right)$ and $\theta = \frac{1}{2}\left(\frac{2\pi}{n}\right) = \frac{\pi}{n}$

Thus the area of the polygon, A (say) is $n\left(\frac{1}{2}\right)\left(\frac{P}{n}\right)\frac{1}{2}\left(\frac{P}{n}\right)/tan\left(\frac{\pi}{n}\right)$

$= \frac{P^2}{4ntan\left(\frac{\pi}{n}\right)}$, as required.

$$\frac{dA}{dn} = \frac{P^2}{4}(-1)\left(ntan\left(\frac{\pi}{n}\right)\right)^{-2}\{tan\left(\frac{\pi}{n}\right) + nsec^2\left(\frac{\pi}{n}\right)(\pi)(-1)n^{-2}\}$$

[The H&A are missing the (-1) near the start.]

We wish to show that $\frac{dA}{dn} > 0$

$\Leftrightarrow tan\left(\frac{\pi}{n}\right) + nsec^2\left(\frac{\pi}{n}\right)(\pi)(-1)n^{-2} < 0 \quad (1)$

$\Leftrightarrow tanx < xsec^2x$, where $x = \frac{\pi}{n}$

$\Leftrightarrow \frac{sinx}{cosx} < \frac{x}{cos^2x}$

$\Leftrightarrow \frac{sinx}{x} < \frac{1}{cosx}$ (1) (as $x > 0$ & $cosx = cos\left(\frac{\pi}{n}\right) > 0$)

[In the H&A, the line which ends with "which tells us that $sinx < x$", should start with ">", rather than "<"]

As $\frac{sinx}{x} < 1$ from (i) and $1 < \frac{1}{cosx}$ for small $x > 0$,

it follows that (1) is true, and hence that $\frac{dA}{dn} > 0$

From Fig. 2, the radius of the circle, r satisfies $rsin\theta = \frac{1}{2}(\frac{P}{n})$

and hence the area of the circle, C (say) is $\pi\left(\frac{(\frac{P}{2n})}{sin(\frac{\pi}{n})}\right)^2 = \frac{xP^2}{4nsin^2x}$

and so $\frac{A}{C} = \frac{\left(\frac{P^2}{4ntanx}\right)}{\left(\frac{xP^2}{4nsin^2x}\right)} = \frac{sin^2xcosx}{xsinx} \approx cosx$, as $\frac{sinx}{x} \approx 1$

So, as $cosx \approx 1$ for small $x = \frac{\pi}{n}$, $\frac{A}{C} \approx 1$, as required.