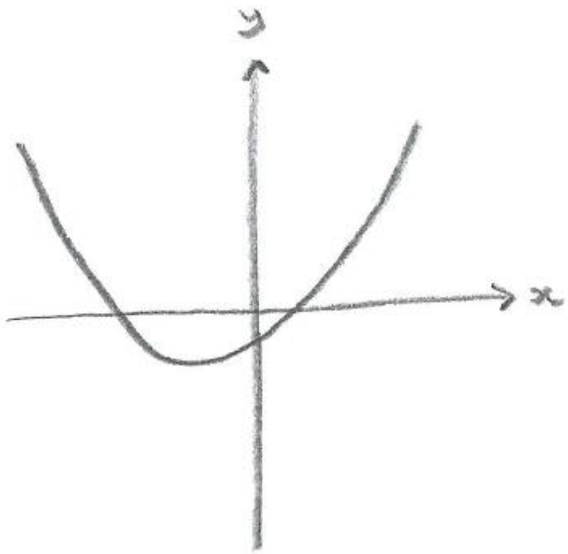
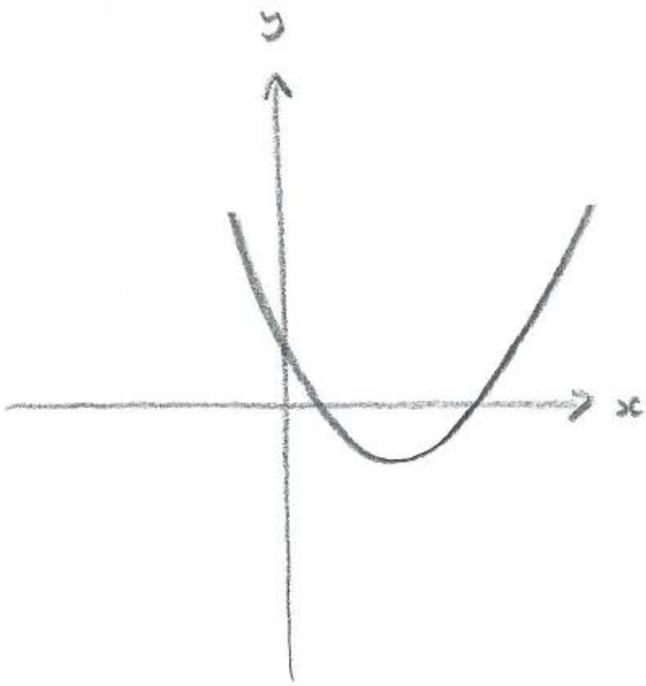
STEP 2006, Paper 1, Q3 - Solution (6 pages; 12/5/18)

(i) If $c < 0$, then the graph of $y = x^2 + bx + c$ will either be as in the diagram below, with the minimum at a negative value of x, or similarly at a positive value of x, or at $x = 0$. In each case, we can see that the graph crosses the x −axis at two distinct points.



As shown in the diagram below, $c < 0$ is not a necessary condition for distinct real roots.



(ii) **Approach 1** (algebraic)

In order for the roots to be real and distinct, we require

$b^2 - 4c > 0$

Then, in order for them both to be positive, we want the smaller root to be positive,

so that $\frac{-b-\sqrt{b^2-4c}}{2} > 0$

and hence $-b > \sqrt{b^2 - 4c} \geq 0$,

so that $b < 0$

Also, in order that $|b| > \sqrt{b^2 - 4c}$, it follows that $c > 0$.

So we have as conditions that are equivalent to the situation of obtaining distinct positive roots that:

$b^2 - 4c > 0,\ \ b < 0\ \ \&\ \ c > 0$

In fact, the first condition means that $|b| > 2\sqrt{c}$, which combines with the 2nd condition to give $b < -2\sqrt{c}$

Thus the conditions reduce to $b < -2\sqrt{c}$ & $c > 0$

[We could in fact replace these with $b < -2\sqrt{c}$ & $c \neq 0$, since $\sqrt{c}$ is undefined for $c < 0$; but $c > 0$ is probably clearer.]

Approach 2 (mainly graphical)

Again, to have distinct real roots, we require that $b^2 - 4c > 0$

Then if $c > 0$, we will either have 2 positive roots, or 2 negative roots (diagrams 1 & 2 below); whereas if $c < 0$, there will be roots of different signs (diagram 3 below). And if $c = 0$, then one of the roots is zero (and so not positive).

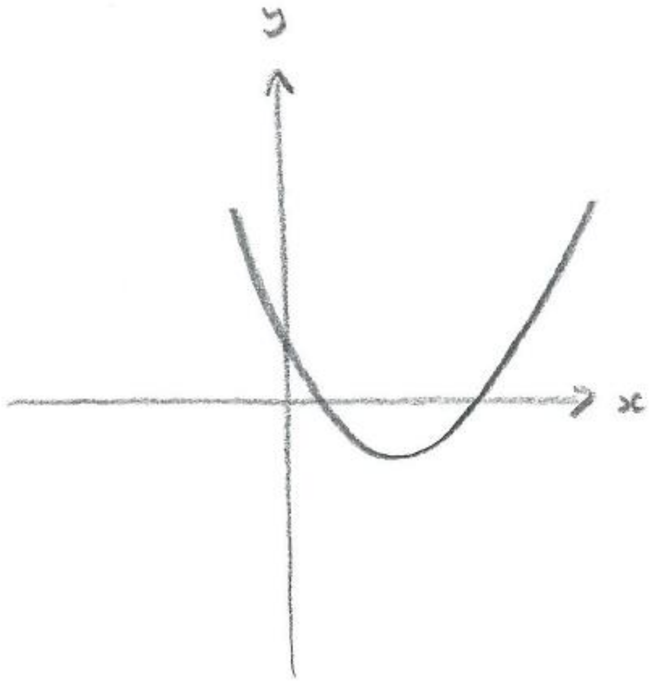


diagram 1

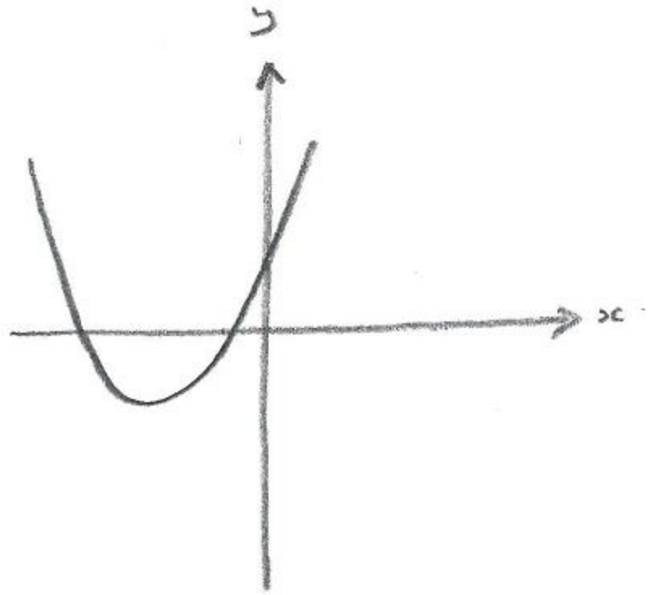


diagram 2

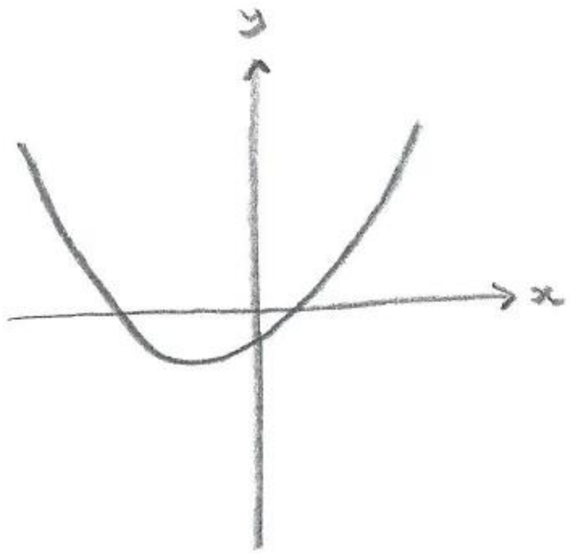


diagram 3

Because the minimum of the curve occurs at $x = -\frac{b}{2a}$ (halfway between the 2 roots), if $b < 0$, together with $c > 0$, then both roots will be positive, whilst if $b > 0$ & $c > 0$, then both roots will be negative.

(iii) Considering the gradient of $y = x^3 + px + q$:

$\frac{dy}{dx} = 3x^2 + p$; so when $p > 0$, $\frac{dy}{dx} > 0$ for all x; ie y is strictly increasing, and so crosses the x-axis once only.

Thus there is 1 real root (and 2 complex roots) of

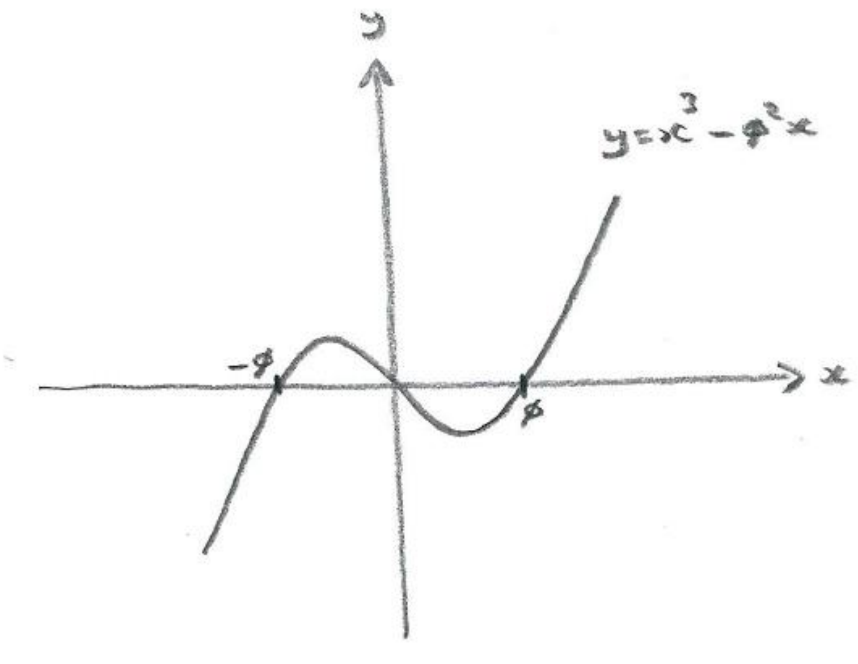
$x^3 + px + q = 0$ when $p > 0$ and $q < 0$.

If $p < 0$ and $q < 0$, let $p = -\phi^2$ (where $\phi > 0$) and consider the simpler graph:

$y = x^3 + px = x^3 - \phi^2 x = x(x - \phi)(x + \phi)$

The number of real roots of $x^3 + px + q = 0$ will then depend on the size of q relative to the height of the local maximum of

$y = x^3 - \phi^2 x$ (see diagram below).



The maximum occurs when $\frac{dy}{dx} = 0$; ie when $3x^2 - \phi^2 = 0$, and $x = -\frac{\phi}{\sqrt{3}}$; when $y = x(x^2 - \phi^2) = -\frac{\phi}{\sqrt{3}}\left(\frac{\phi^2}{3} - \phi^2\right) = \frac{2\phi^3}{3\sqrt{3}}$

So, when $|q| < \frac{2\phi^3}{3\sqrt{3}}$, there will be 3 distinct real roots.

When $|q| = \frac{2\phi^3}{3\sqrt{3}}$, there will be 3 real roots, of which 2 are repeated.

When $|q| > \frac{2\phi^3}{3\sqrt{3}}$, there will be 1 real root (and 2 complex roots).

Now, $4p^3 + 27q^2 > 0 \Leftrightarrow 27q^2 > -4p^3 \Leftrightarrow 3\sqrt{3}|q| > 2\phi^3$

$\Leftrightarrow |q| > \frac{2\phi^3}{3\sqrt{3}}$, and similarly for the other cases.

So, if (a) $4p^3 + 27q^2 > 0$, there will be 1 real root (and 2 complex roots);

if (b) $4p^3 + 27q^2 = 0$, there will be 3 real roots, of which 2 are repeated,

and if (c) $4p^3 + 27q^2 < 0$, there will be 3 distinct real roots.

From the above diagram, we can also comment on the signs of the roots:

For (a), the root is positive.

For (b), the repeated root is negative and the other one is positive.

For (c), 2 of the roots are negative and the other is positive.